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"Analytical Geometry
of three dimensions" Salmon.

Lec. on 11th Oct. '21 ———— . . . . ———— 13th Oct. 1921.

Co-ordinates:—

The co-ordinates of any point in space can be represented by, as we know

(i) x, y, z

(ii) r, θ, ϕ

(iii) r, θ, z

Any equation of the form,

$$f(x,y,z) \equiv \phi(r,\theta,\phi) \equiv F(r,\theta,z) = 0$$

represents a surface.

If after solving for z we get

$z = f(x,y)$, we see that by altering x, or y we get different values for z.

Thus a Surface consists of two-fold infinity of points.

A surface can be represented also by $x = f_1(u,v)$; $y = f_2(u,v)$; $z = f_3(u,v)$

P. T. O

where u, v are called the parameters.

A curve in general is represented by two equations in x, y, z. — the intersections of two surfaces being curves. e.g.

$$f(x, y, z) = 0$$
$$\phi(x, y, z) = 0$$

A curve consists of single infinity of points. It can be represented also by:—

$$x = f_1(t),$$
$$y = f_2(t),$$
$$z = f_3(t).$$

where t is the parameter

// Returning to the equations of a surface in the form

$$x = f_1(u, v)$$
$$y = f_2(u, v)$$
$$\& \; z = f_3(u, v)$$

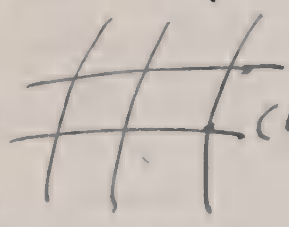
we see that u = constant, the

equations represent a curve, v taking place of t.

$$\therefore\ u = \text{Constant},$$
represents curves.

Similarly $v = \text{Constant}$
represents a family of curves.

A surface, therefore is said to be <u>striated</u> by two families of curves.



An Example:— A rod of given length has its ends on two surfaces say $f(x, y, z) = 0$
& $\phi(x, y, z) = 0$

To find the positions of its mid-pt.

If (x_1, y_1, z_1) $(x_2\ y_2\ z_2)$ are the extremities & (x, y, z) the mid pt. we have got—

P. T. O.

the following six equations:-

$$x = \tfrac{1}{2}(x_1 + x_2)$$
$$y = \tfrac{1}{2}(y_1 + y_2)$$
$$z = \tfrac{1}{2}(z_1 + z_2)$$
$$f(x_1, y_1, z_1) = 0$$
$$\& \ \phi(x_2, y_2, z_2) = 0$$
$$\& \ l^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$$

And the variables are six in number viz $x_1\ x_2\ x_3\ y_1\ y_2\ y_3$.

It appears then at first sight that the mid-pt. can occupy any position in space.

But that it cannot is obvious. What is the explanation?

What happens is this: In general the centre of the rod may occupy any position within a certain region. And the boundary of this sol[i]d is determined by the condition of the

co-ordinates being real.

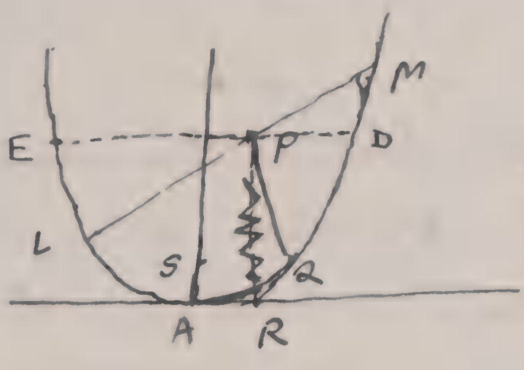
Suppose for example:—

length of the rod $= 2c$

and it slides inside a paraboloid of revolution, whose equation is

(i)

$$r^2 = x^2 + y^2 = 4az.$$



$P\,(x, y, z)$. 15th Dec 1912

[fig is not correct. PQ being a diameter is $\parallel AS$]

LM is the major axis of the elliptic section through P $\parallel$l to the tangent plane at Q.

Obviously the length of the rod must lie between the minor & major axes of this ellipse.

Let α = Semimajor axis $= PM = c$

& β = semi-minor axis, which is $\perp$ plane of the paper & through P

Now by the property of parabola

$$\alpha^2 = PM^2 = 4SQ \cdot PQ$$

$$= 4(a + z') \cdot (z - z')$$

$$= 4\left(a + \frac{r^2}{4a}\right)\left(z - \frac{r^2}{4a}\right)$$

$$= 4(4a^2 + x^2 + y^2)(4az - x^2 - y^2)/16a^2$$

$$\therefore 4c^2a^2 = (4a^2 + x^2 + y^2)(4az - x^2 - y^2) \quad \dots \quad \mathrm{I}$$

this is one boundary.

Similarly, if we were to take a parallel circular section through P, the minor axis will be the common section of this circle and the ellipse.

$$\therefore \beta^2 = PD \cdot PE$$

$$= -PF^2 + EF^2$$

$$= -r^2 + 4az$$

$$\therefore c^2 = 4az - x^2 - y^2 \quad \dots \quad \mathrm{II}$$

this is the second boundary.

(ii)

From the I equation we get

$$4a z_1 = \frac{4c^2a^2}{4a^2+r^2} + x_1^2 + y_1^2$$

& from the II

$$4a z_2 = c^2 + x_1^2 + y^2$$

$$\therefore z_1 - z_2 = \frac{c^2 r^2}{4a(4a^2+r^2)} = +\text{ve quantity}$$

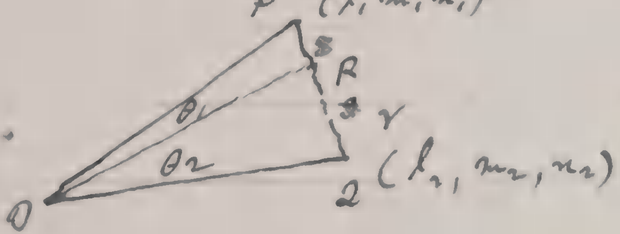
Therefore I represents the upper boundary
& II ,, ,, lower boundary.

It is to be noted, then, that the locus of the mid-point is a <u>solid</u>.

Q.E.D.

"Direction Cosines"

If a <u>unit</u> sphere be described with the origin as centre, then the co-ordinates of the ends of radii are the direction cosines of the radii.



Let R divide PQ in the ratio r/s

The co-ordinates of R are

$$\left(\frac{rl_1 + sl_2}{r+s} ;\ \frac{rm_1 + sm_2}{r+s} ;\ rk_3\ \frac{rn_1 + sn_2}{r+s} \right)$$

Also $s/r = \sin\theta_1 / \sin\theta_2$

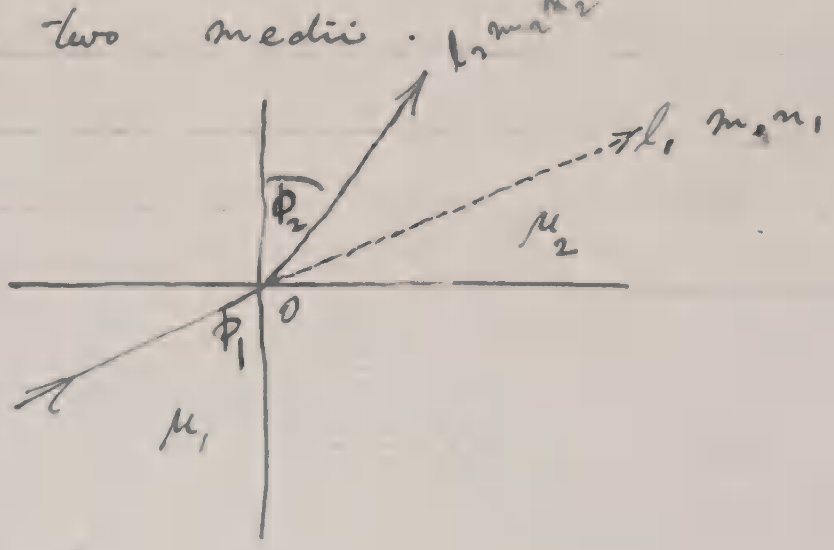
We have thus divided PQ in the ratio $\sin\theta_1 : \sin\theta_2$.

Consider the following illustration:- [Optics]

We know that the law of refraction is that, when a ray passes from one medium into another, the angles of incidence in them are connected by the relation

$$\mu_1 \sin\phi_1 = \mu_2 \sin\phi_2$$

$\mu_1\ \mu_2$ being the indexes of refraction for the two media.



& we get

$$\frac{\mu_2 l_2 - \mu_1 l_1}{\lambda} = \frac{\mu_2 m_2 - \mu_1 m_1}{\mu} = \frac{\mu_2 n_2 - \mu_1 n_1}{\nu}$$

where,

$$l_1 \lambda + m_2 \mu + n_3 \nu = \cos\phi_1$$

$$l_2 \lambda + m_2 \mu + n_2 \nu = \cos\phi_2.$$

[λ μ ν are the direction cosines of the normal to the incident plane.]

"The Plane"

A plane may be obtained by three non-collinear points, — $(x_r\ y_r\ z_r)$ $r = 1, 2, 3$.

The centre of mass of masses, m_r at $(x_r\ y_r\ z_r)$ $r = 1, 2, 3$ is

$$x = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3}; \ \&c$$

Evidently the centre of mass lies in the plane.

$\therefore$ The co-ordinates of any pt. in the plane can be expressed as

$$x = \frac{lx + my + nz}{l + m + n}; \ \&c. \quad (?)$$

[Deduce the equation to such a plane]

Different formulae for the volume of a tetrahedron:—

(i) $$V = \frac{1}{6} \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}.$$

$$= \frac{1}{6} aa' (\overline{a, a'}) \sin \widehat{a a'}.$$

$$= \frac{1}{6} abc \sqrt{1 - \cos^2\lambda - \cos^2\mu - \cos^2\nu + 2\cos\lambda\cos\mu\cos\nu}.$$

= in terms of six sides

= in terms of four planes. &c.

Lect. on 13th Oct. 1926 13th Oct. 1926.

"Straight Line"

A st. line can be represented by two equations,

$$y = mx + n$$
$$z = px + q$$

these being the projections on two Co-ordinates planes.

Thus there are four independent Constants.

which means that there are four-fould infinity of st. lines in Space.

In treating of ruled surface we might use any two equations of a st. line on the assumption that the four Constants are functions of a variable parameter (t)

The Co-ordinate of the Surface in that case can be expressed in terms of x & t.

"Six Co-ordinates"

The symmetrical form of the eqn. to a st. line viz.

$$\frac{x-f}{l} = \frac{y-g}{m} = \frac{z-h}{n}$$

25th Dec. 1922

$$\left.\begin{aligned} \equiv \quad & yn - mz = ng - mh \equiv \lambda \\ & zl - nx = lh - nf \equiv \mu \\ \& \quad & xm - yl = fm - gl \equiv \nu \end{aligned}\right\| \quad \lambda, \mu, \nu \equiv \begin{Vmatrix} x & y & z \\ l & m & n \end{Vmatrix}$$

Any two of these equations define a st. line, the third being capable of deduction.

The six quantities $\lambda, \mu, \nu, l, m, n$ are called the six co-ordinates of the line.

[l, m, n may not be the actual direction cosines]

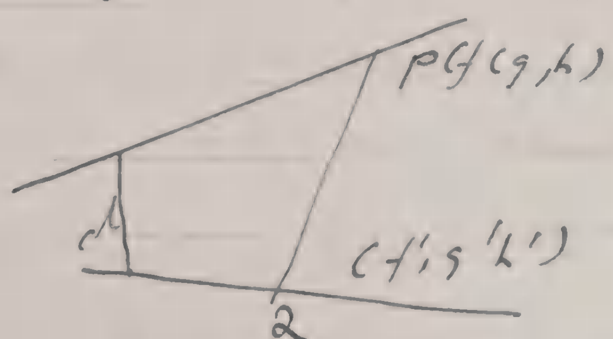
It is only their ratios that define the line, & they satisfy identically the relation, $l\lambda + m\mu + n\nu = 0$

The five ratios with this condition are equivalent to four arbitrary constants.

If we take a unit force in the st. line, & suppose l, m, n to be the <u>actual</u> direction cosines also then, l, m, n λ, μ, ν are the components respectively of the force at the origin & the couple — which are both together equal to given unit force.

Suppose now we take two st. lines

$$\frac{x-f}{l} = \quad = \frac{z-h}{l}$$
$$\frac{x-f'}{l'} = \quad = \frac{z-h'}{n'}.$$



we see that they will intersect if PQ & the two lines are all

perpendicular to a common line,

i.e. if $\begin{vmatrix} f-f', & g-g', & h-h' \\ l & m & n \\ l' & m' & n' \end{vmatrix} = 0$

i.e. $\begin{vmatrix} f & g & h \\ l & m & n \\ l' & m' & n' \end{vmatrix} - \begin{vmatrix} f' & g' & h' \\ l & m & n \\ l' & m' & n' \end{vmatrix} = 0$

i.e. $\Sigma\, l'(mh - gn) - \Sigma\, l'(mh' - g'n) = 0$

i.e. $\Sigma - l'\lambda + \Sigma\, l(m'h' - g'n') = 0$

i.e. $\Sigma - l'\lambda + \Sigma - l\lambda' = 0$

i.e. $l'\lambda + m'\mu + n'\nu + l\lambda' + m\mu' + n\nu' = 0$

which is the same co-ordinate condition of the lines intersecting.

When they do not intersect, it is equal to the product of

$(\overline{a, a'}) \sin(\widehat{a, a'})$.

where a, a' are the two lines.

It is also then equal to

the value of the moment of a unit force along either of the lines with the other.

"Transformation of Axes":—

Rectangular:—

If $(l_r\ m_r\ n_r)$ $r = 1, 2, 3$ be the direction cosines of the 2nd set of axes; then the transformation is given by the following Scheme:—

	x	y	z
x'	l_1	m_1	n_1
y'	l_2	m_2	n_2
z'	l_3	m_3	n_3

$$x' = l_1 x + m_1 y + n_1 z \quad ; \&c$$

$$x = l_1 x' + l_2 y' + l_3 z' \; ; \&c.$$

To consider the determinant more closely

$$\Delta \equiv \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$$

We shall find that the follow

are its properties :-

(1) The sum of the sqs. of elements of any row or any column $= 1$

(2) Product of the corresponding elements of any two rows or columns $= 0$

(3) $\Delta = \pm 1$

(4) And each element of $\Delta =$

$=$ its minor with the + or -ve sign acc. as Δ is +ve or -ve.

We have by (2),

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$$

$$l_1 l_3 + m_1 m_3 + n_1 n_3 = 0$$

$$\therefore \frac{l_1}{m_2 n_3 - m_3 n_2} = \frac{m_1}{\quad} = \frac{n_1}{\quad}$$

$$= \frac{l_1^2 + m_1^2 + n_1^2}{\Delta} = \frac{1}{\Delta}$$

Again each $= \dfrac{\Sigma l_1 (m_2 n_3 - m_3 n_2)}{\Sigma (m_2 n_3 - m_3 n_2)^2}$

$$= \frac{\Delta}{(m_1^2 + m_2^2 + m_3^2)(n_1^2 + n_2^2 + n_3^2) - - \{\ \}} = \frac{\Delta}{1}$$

$$\therefore \frac{1}{\Delta} = \frac{\Delta}{1}$$

$$\text{or } \Delta^2 = 1 \quad \text{i.e. } \Delta = \pm 1$$

&c &c.

If the minors be as usual denoted by L_1, M_1, N_1 &c.

we get

$$\frac{l_1}{L_1} = \frac{m_1}{M_1} = \frac{n_1}{N_1} = \frac{l_r}{L_r} = \frac{1}{\Delta}.$$

or, we may briefly put it thus

$$\frac{L_r}{l_r} = \frac{M_r}{m_r} = \frac{N_r}{n_r} = \Delta = \pm 1$$

$$(r = 1, 2, 3).$$

Q.E.D.

It is worthy of note that the distinction between the two cases when Δ is +ve & −ve occurs thus:—

Δ is +ve when both systems are of the same type — both left-handed or both right-handed.

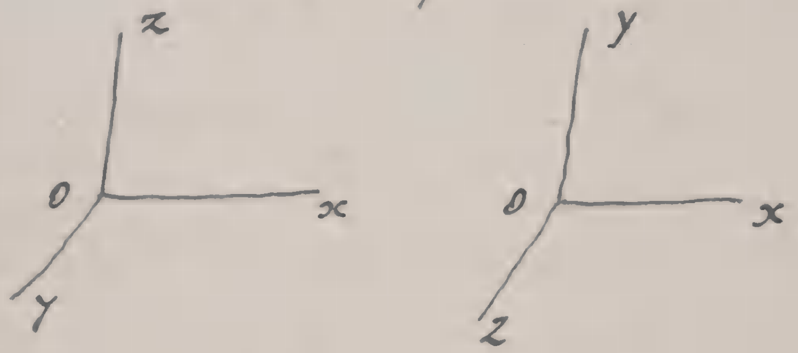
otherwise Δ is −ve.

When both systems are of the same type it is a case of real rotation — i.e. the first set of axes can be made to

Coincide with the other by simple rotation-process.

In the 2nd cases it is impossible to make the axes coincide.

We represent the distinction :-



There are systems of different types.

To work out the following Examples :-

o o o o

∴ The Condition that $(l\,m\,n\,\lambda\,\mu\,\nu)$ & $(l'm'n'\lambda'\mu'\nu')$ should intersect can be got as follows :-

The unit force along OP has no moment about OQ which gives

$\Sigma l\lambda' + \Sigma l'\lambda = 0$. (25th Dec. 1922.)

Ex. 2
p. 50

To prove that if r is the radius of the circumscribed ~~circle~~ sphere of a tetrahedron then,

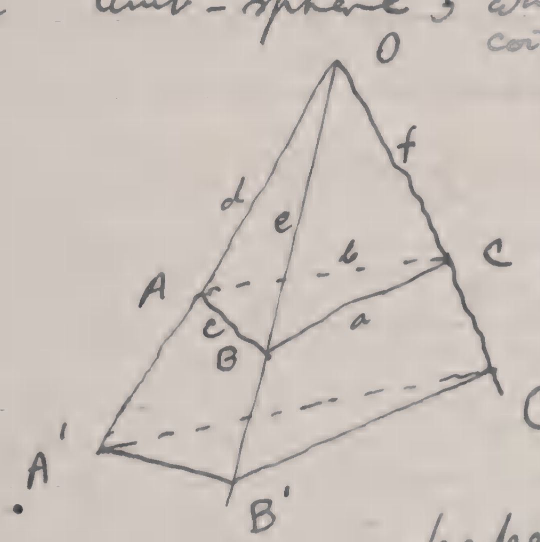
$$6rV = \sqrt{S(S-ad)(S-be)(S-cf)}$$

where $2S = ad + be + cf$

& a, b, c, d, e, f the six edges of the tetrahedron.

Proof:- We shall do it by the method of Inversion.

Invert the tetrahedron by means of the vertex O with regard to the unit-sphere, whose diameter through O coincides with that of the circum-sphere (?)



Then A', B', C' the inverses of A, B, C respectively will lie in a plane the perpendicular on which from O

$= \frac{1}{2r}$.

Obviously, $OA' = \frac{1}{OA} = \frac{1}{d}$

$OB' = \frac{1}{e}$; & $OC' = \frac{1}{f}$.

$$C'B'^2 = OB'^2 + OC'^2 - 2\,OB'\cdot OC' \cos \widehat{OB', OC'}$$

$$= \frac{1}{e^2} + \frac{1}{f^2} - 2\frac{1}{ef}\cos OBC$$

$$= \frac{e^2 + f^2 - 2ef\cos OBC}{e^2 f^2}$$

$$= \frac{a^2}{e^2 f^2}$$

$$\therefore \quad B'C' = a/ef$$

$$C'A' = b/fd \;;\; \text{\&} \;\; A'B' = c/de$$

Now if V' is the volume of $(O, A'B'C')$

$$\tfrac{1}{3}\, p \cdot \Delta A'B'C' = V'$$

i.e. $$\frac{1}{6r}\, \Delta A'B'C' = V'$$

i.e. $$6rV' = \Delta A'B'C'$$

$$= \sqrt{s(s-ad)(s-be)(s-cf)} \Big/ d^2e^2f^2$$

Since a/ef, b/fd, c/de are the sides.

· Now we know that in two tetrahedra which have a common set three conterminous edges.

their volumes are proportional to the product of those three edges;—

Thus $\quad V : V' = def : 1/def$

$$\therefore \quad V' = \frac{V}{d^2e^2f^2}$$

Substituting this value we get at once

$$6rV = \sqrt{S(S-ad)(S-be)(S-cf)}$$

Q.E.D.

We proceed to work out the 2nd example which was set :—

To find ~~the equation of the~~ ~~mean~~ the value of the angle between the two lines determined by

$$\phi \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \quad \cdots (1$$

$$\& \quad ux + vy + wz = 0 \quad \cdots \cdots (2$$

Suppose, (l m n) are the direction Cosines of a line of intersection

Then l, m, n will have two sets of values

Since the line lies in (1 & (2

$$\phi(l, m, n) = 0 \quad \& \quad ul + vm + wn = 0$$

eliminating say n

i.e. putting $n = -\dfrac{ul + vm}{\omega}$ in (i)

we get

$$al^2 + bm^2 + c\,\frac{(ul+vm)^2}{\omega^2} + 2fm\cdot\frac{-(ul+vm)}{\omega}$$

$$+ 2gl\cdot\frac{-(ul+vm)}{\omega}$$

$$+ 2hml \quad = 0$$

i.e.

$$l^2\left(a + \frac{cu^2}{\omega^2} - \frac{2gu}{\omega}\right) + 2lm\left(\frac{cuv}{\omega^2} - \frac{uf}{\omega} + h - \frac{gv}{\omega}\right)$$

$$+ m^2\left(b + \frac{cv^2}{\omega^2} - \frac{2fv}{\omega}\right) \quad = 0$$

i.e. $$l^2\left(a\omega^2 + cu^2 - 2gu\omega\right) - 2lm\left(fu\omega + gv\omega - cuv - h\omega^2\right)$$

$$+ m^2\left(b\omega^2 + cv^2 - 2fv\omega\right) = 0$$

This determines two ratios of l to m

i.e. $\dfrac{l_1}{m_1}$ $\quad \dfrac{l_2}{m_2}$

& we get

$$\frac{l_1 l_2}{b\omega^2 + cv^2 - 2fv\omega} = \frac{m_1 m_2}{a\omega^2 + cu^2 - 2gu\omega} = \frac{l_1 m_2 + l_2 m_1}{2(fu\omega + gv\omega - cuv - h\omega^2)}$$

$$= \sqrt{\frac{(l_1 m_2 + l_2 m_1)^2 - 4 l_1 l_2 m_1 m_2}{- \quad . \quad -}} \quad \underline{\qquad\qquad}$$

$$= \pm \frac{l_1 m_2 - l_2 m_1 \qquad\qquad}{2\sqrt{(fuw + gvw - cuv - hw)^2 - (bw^2 + - -)(aw^2 + -)}}$$

Similarly each of these ratios =

$$\frac{n_1 n_2}{av^2 + bu^2 - 2huv} = \pm \frac{l_1 n_2 - l_2 n_1}{2\sqrt{\quad}} = \pm \frac{n_1 m_2 - n_2 m_1}{2\sqrt{\quad}}$$

each ratio is again =

$$\frac{\Sigma l_1 l_2}{\qquad\qquad} = \sqrt{\frac{\Sigma (l_1 n_2 - l_2 n_1)^2}{4(\sqrt{- \ .}\ )^2}}$$

$$= \frac{\cos\theta}{\Sigma a(v^2 + w^2) - \Sigma 2fvw} = \frac{\sin\theta}{2P}$$

i.e. $\cancel{\tan\theta}$ $\cot\theta$ $= \dfrac{(a+b+c)(u^2+v^2+w^2) - \phi(u, v, w)}{2P}$

from this we at once get,

If the lines are mutually perpendicular

$$\cos\theta = 0$$

$$\& \quad (a+b+c)(u^2+v^2+w^2) = \phi(u,v,w) \quad \cdots \quad I$$

Secondly, if the lines are coincident

$$\sin\theta = 0$$

$$\text{i.e.} \quad \rho \equiv 0$$

The value of ρ^2 is of course

$$(fuw + gvw - cuv - hw^2)^2 - (bw^2+cv^2-2fvw)(aw^2+cu^2-2guw)$$

This when solved will be found to be a symmetrical expression in $(f\,g\,h \quad u\,v\,w)$.

<u>Note</u> :- In fact this example has been treated as an Article in Bell's Solid Geometry.

Equation (i) ordinarily represents a cone with vertex at the origin.

Ex. 3 / p. 50 The shortest distance between two opposite edges a, d of a tetrahedron is

$$\frac{6V}{ad\sin\theta} \quad \text{where } \theta = \widehat{a,d}.$$

Let the equations of lines along which a & d lie be

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r$$

$$\frac{x-\alpha'}{l'} = \frac{y-\beta'}{m'} = \frac{z-\gamma'}{n'} = r'$$

If the first extremities of the two lines be $P(\alpha, \beta, \gamma)$

& $R(\alpha' \beta' \gamma')$

Then, the other extremities are

$$Q(\alpha + al, \beta + am, \gamma + an)$$

& $S(\alpha' + dl', \beta' + dm', \gamma' + dn')$

The volume of the tetrahedron whose vertices are known is V; where

$$6V = \begin{vmatrix} \alpha + al, & \beta + am, & \gamma + an & 1 \\ \alpha' + dl', & \beta' + dm', & \gamma' + dn' & 1 \\ \alpha & \beta & \gamma & 1 \\ \alpha' & \beta' & \gamma' & 1 \end{vmatrix}$$

which gives

$$6V = \begin{vmatrix} al, & am, & an & 0 \\ dl', & dm', & dn' & 0 \\ \alpha & \beta & \gamma & 1 \\ \alpha' & \beta' & \gamma' & 1 \end{vmatrix}$$

$$= -ad \begin{vmatrix} l, & m, & n \\ l', & m', & n' \\ \alpha - \alpha' & \beta - \beta' & \gamma - \gamma' \end{vmatrix}$$

Now, the shortest distance

$$= \frac{|\quad\quad|}{\cancel{x}\, S \sin\theta} = |\quad|\, S \sin\theta$$

$$\therefore\ 6V = ad\,\bar{s} \sin\theta$$

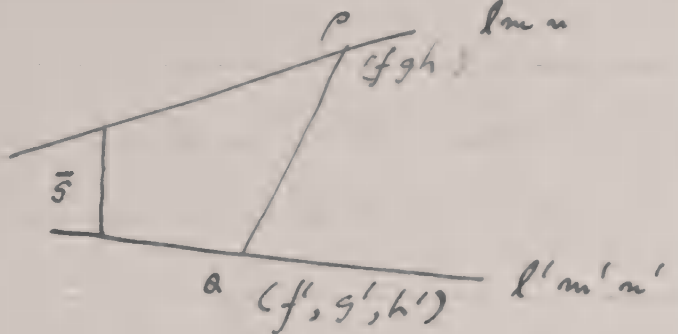
$$\therefore\ \bar{s} = \text{shortest distance}$$

$$= \frac{6V}{ad \sin\theta}.$$

Note:— If we examine the determinant rather more closely

$$\Delta = \begin{vmatrix} \alpha - \alpha' & \beta - \beta' & \gamma - \gamma' \\ l & m & n \\ l' & m' & n' \end{vmatrix}.$$

where the symbols are—



$\bar{s}$ = projn. of PQ on $\bar{s}$ which is
⊥ both lines.

the directions of $\bar{s}$ are
proportional to
$mn' - m'n, \; nl' - n'l, \; lm' - l'm$.

$$\therefore \; \bar{s} = \frac{(f-f')\lambda + (g-g')\mu + \nu(h-h')}{\sqrt{\Sigma\,(mn'-m'n)^2}}$$

$$= \frac{\left|\quad\right|}{\sin^2\theta}$$

$$\therefore \; \bar{s}\,\sin\theta = \Delta .$$

Q.E.D.

Lect. on 15th Oct. 1921 16th Oct. 1921.

Eulerian Transformation of Co-ordinates:—

The following method, due to Euler can bring one set of rectangular axes into any other position. It consists of three rotations:—

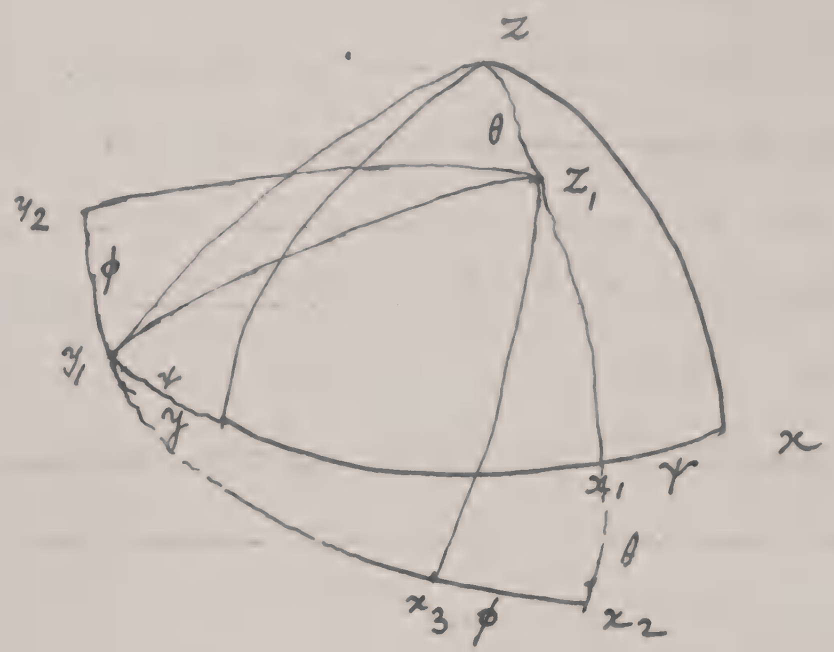
(i) About the axis of z, through ψ

(ii) About the new axis of y, through θ

& (iii) lastly about the new axis of z through an angle ϕ.

By choosing θ, ϕ, ψ suitably any transformation can be effected, in which the origin remains unaltered & the system remains of the same type — i.e. either left-handed or right-handed.

The following fig. will serve to illustrate:—



$x\,y\,z$ - - - → Original Set.

$x_1\,y_1\,z$ - - - → After rotation about Oz through ψ,

$x_2\,y_1\,z_1$ ——→ After rotation about Oy_1 through θ,

$x_3\,y_2\,z_1$ —→ After ,, ,, Oz_1, through φ.

— the new set.

(Thus we begin with z-axis & end with the new z-axis)

To find the direction Cosines of the new axes with respect to the old ones :—

[Let $P(\bar{l})$ ≡ vector or force of l units magnitude acting along OP, O being the centre of the sphere].

$$x_3(T) \equiv x_2(\overline{\cos\phi}) + y_1(\overline{\sin\phi})$$

$$\equiv y_1(\overline{\sin\phi}) + x_1(\overline{\cos\phi\cos\theta}) - z(\cos\phi\sin\theta).$$

$\therefore$ Resolving the system along the old axes, Ox, Oy, Oz, we get

$$\left.\begin{aligned} l_1 &= \cos\theta\cos\phi\cos\psi - \sin\phi\sin\psi. \\ m_1 &= \sin\phi\ \cancel{\cos}\ \psi + \cos\phi\cos\theta\sin\psi. \\ n_1 &= -\cos\phi\sin\theta. \end{aligned}\right\}.$$

Again,

$$y_2(T) \equiv y_1(\overline{\cos\phi}) - x_2(\overline{\sin\phi}).$$

$$\equiv y_1(\overline{\cos\phi}) - x_1(\sin\phi\cos\theta) + z(\sin\phi\sin\theta)$$

Resolving as before

$$\therefore \left.\begin{aligned} l_2 &= -\cos\phi\sin\psi - \sin\phi\cos\theta\cos\psi. \\ m_2 &= \cos\phi\cos\psi - \sin\phi\cos\theta\sin\psi \\ n_2 &= \sin\phi\ \cancel{\sin\theta} \end{aligned}\right\}.$$

Similarly, we get

P.T.O.

$$z_1(T) \equiv z(\cos\theta) + x_1(\sin\theta)$$

∴ Resolving as in the preceding two cases we have,

$$\left.\begin{aligned} l_3 &= \sin\theta\cos\psi \\ m_3 &= \sin\theta\sin\psi \\ \text{\&}\ n_3 &= \cos\theta. \end{aligned}\right\}$$

Thus the transformation determinant

$$\Delta \equiv \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$$ becomes in Eulerian letters:-

$$\begin{vmatrix} \cos\theta\cos\phi\cos\psi - \sin\phi\sin\psi, & \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi, & -\cos\phi\sin\theta \\ -\cos\phi\sin\psi - \sin\phi\cos\theta\cos\psi, & \cos\phi\cos\psi - \sin\phi\cos\theta\sin\psi, & \sin\phi\sin\theta \\ \sin\theta\cos\psi, & \sin\theta\sin\psi, & \cos\theta \end{vmatrix}$$

It can be verified that $\Delta \equiv 1$ & each element = its co-factor in the determinant.

Thus for instance, the co-factor of the 1st term which corresponds to l_1 is

$$= (\cos\phi\cos\psi - \sin\phi\cos\theta\sin\psi)\cos\theta$$
$$- \sin^2\theta\sin\psi\sin\phi$$

$$= \cos\phi\cos\psi\cos\theta - \sin\phi\sin\psi(\sin^2\theta + \cos^2\theta)$$

$$= \cos\phi\cos\theta\cos\psi - \sin\phi\sin\psi$$

$$= l_1 \text{ itself.}$$

And similarly for others.

$$\therefore \Delta \equiv l_1L_1 + m_1M_1 + n_1N_1$$

$$\equiv l_1^2 + m_1^2 + n_1^2$$

$$\equiv +1$$

Q.E.D.

Note:— There is however no symmetry in the results, & no elegance. It is rather strange why it should be called a method. One point however; Just as in a plane any point can be transferred to any other by two motions of translation parallel to the axes

Similarly, in space the axes can be rotated into any position by three rotatory motions in the co-ordinate planes.

"Surfaces of the Second Degree"

The most General Equation of the second degree is

$$\phi(x,y,z) \equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy$$
$$+2ux+2vy+2wz+d=0$$

If we wish to make it homogeneous we can introduce a letter t $(\equiv 1)$. & write in the form,

$$\phi(x,y,z,t) \equiv ax^2+ \dots +2t(ux+vy+wz)+dt^2=0$$

General Propositions:-

Any point on the line joining the points $P(x_1 y_1 z_1)$ $Q(x_2 y_2 z_2)$ is

$$\left(\frac{x_1+\lambda x_2}{1+\lambda}, \ \&c., \quad\right).$$

In order to obtain the points of intersection of the line PQ with $\phi=0$ we can substitute & get a quadratic in λ

$$\lambda^2 U_2 + 2\lambda U_{12} + U_1 = 0$$

where $U_2 = \phi(x_2 y_2 z_2)$
$U_1 = \phi(x_1 y_1 z_1)$ &

$$U_{12} \equiv \tfrac{1}{2}\left(x_1 \frac{\partial u_2}{\partial x_2} + y_1 \frac{\partial u_2}{\partial y_2} + z_1 \frac{\partial u_2}{\partial z_2} + t_1 \frac{\partial u_2}{\partial t_2}\right)$$

$t_1 = t_2 = 1$

(i) If now $P(x, y, z)$ lies on the surface $U_1 = 0$ & one of the roots of the Quadratic $= 0$

The second also will be equal to zero if $U_{12} = 0$

$\therefore$ $U_{12} \equiv 0$ is the condition of the line being a tangent.

And since the U_{12} is linear in $x_2\, y_2\, z_2$, the locus of tangent lines at P is a plane — the tangent plane.

(ii) If, however, $P(x, y, z)$ does not lie on the surface, the line will be divided harmonically i.e. in the same ratio by the points of section if

$$\lambda_1 = -\lambda_2$$

$$\text{or } \lambda_1 + \lambda_2 = 0$$

i.e. again $U_{12} = 0$

In this case, writing u_{12}, xyz for $x_2 y_2 z_2$ we get a plane which is the locus of the inverse of $(x_1 y_1 z_1)$

The plane is called the polar plane.

[general theory in Edwards Cal]

Lect. on 18th Oct. 1921 19th Oct. 1921
8-(a.m.)

(iii) Returning to the standard equation in λ dealt with last time viz:—

$$\lambda^2 U_2 + 2\lambda P_{12} + U_1 = 0$$

The quadratic might vanish altogether whatever λ was, — when

$$U_1 \equiv U_2 \equiv P_{12} \equiv 0$$

Which means the original points $(x_1 y_1 z_1)$ & $(x_2 y_2 z_2)$ are not only on the surface but are such that each one is on the polar plane of the other. And the whole line joining them lies on the surface.

All surfaces of the second degree admit of st. lines being drawn on them which are however real under special conditions.

Again the condition for equal roots is $U_1 U_2 = P_{12}^2$, which is the condition for the line touching the conic.

If, therefore, we replace $x_2 y_2 z_2$ by xyz we get $U_1 U = P_1^2$ as the equation of all

lines that can be drawn to touch from the point x_1, y_1, z_1. Hence, it is the equation of the enveloping cone.

The points of contact are on the polar plane evidently.

(iv) Let us take the plane

$$lx + my + nz + p = 0$$

and identify it with the polar plane of a point $P_1 = 0$

$$P_1 \equiv axx_1 + byy_1 + czz_1 + g(zx_1 + xz_1) + f(yz_1 + zy_1) + h(xy_1 + yx_1) + u(x + x_1) + v(y + y') + w(z + z_1) + d = 0$$

We get then

$$\begin{aligned} ax_1 + hy_1 + gz_1 + u &= \mu l \\ hx_1 + by_1 + fz_1 + v &= \mu m \\ gx_1 + fy_1 + cz_1 + w &= \mu n \\ ux_1 + vy_1 + wz_1 + d &= \mu p. \end{aligned}$$

These are four equations in x_1, y_1, z_1 & we can solve them by introducing a fourth co-ordinate t $(=1)$.

And we shall get:—

$$x_1 \Delta \equiv \mu(lA + mH + nG + U)$$
$$y_1 \Delta \equiv \mu(lH + mB + nF + V)$$
$$z_1 \Delta \equiv \mu(lG + mF + nC + W)$$
$$t_1 \Delta \equiv \mu(lU + mV + nW + D).$$

The capital letters are the elements of the reciprocal determinant i.e. the first minors with the correct sign.

The condition of tangency is that (x_1, y_1, z_1) must lie on the surface i.e.

$$(A\,B\,C\,D\,E\,F\,G\,H\,U\,V\,W \between l\,m\,n\,p)^2 = 0$$

Which, therefore the tangential equation of the conicoid.

The determination of the polar plane evidently breaks down if $\Delta \equiv 0$; & we fail to find the tangential equation.

The vanishing of Δ represents

the surface represents a cone, & the reason why the process fails is that cone is a different kind of surface.

The general Conicoid has a double infinity of tangent planes, while the cone only a single one.

To write the tangential equation of the cone two conditions are required.

Lect. on 20th Oct. 1921 — 21st. Oct. 1921.

The centre or centres of the Quadric.

If the equation is satisfied by $(x\,y\,z)$ when referred to the centre as the origin then $(-x, -y, -z)$ also must satisfy.

Hence the form of the equation is

$$u_2 + d' = 0$$

instead of $u_2 + u_1 + d = 0$

Let (α, β, γ) be the centre of $\phi(x, y, z) = 0$

Then the equation when referred to centre is

$$\phi(x+\alpha, y+\beta, z+\gamma) = 0$$

And remembering that $u_1 \equiv 0$, by equating the coefficients x, y, z separately to zero we get.

$$\phi(x+\alpha, y+\beta, z+\gamma) \equiv \phi(\alpha, \beta, \gamma)$$

$$+ \left(x\frac{\partial}{\partial\alpha} + y\frac{\partial}{\partial\beta} + z\frac{\partial}{\partial\gamma}\right)\phi$$

$$+ \frac{1}{2!}\left(x\frac{\partial}{\partial\alpha} + y\frac{\partial}{\partial\beta} + z\frac{\partial}{\partial\gamma}\right)^2\phi$$

$$\equiv \phi(\alpha, \beta, \gamma) + u_2.$$

$\therefore$ The equation as referred to centre is $\quad u_2 + \phi(\alpha, \beta, \gamma) = 0$

Now we have made coefficients of x, y, z zero. that

$$\left.\begin{aligned}
\tfrac{1}{2}\frac{\partial \phi}{\partial \alpha} &\equiv a\alpha + h\beta + g\gamma + u = 0 \\
\tfrac{1}{2}\frac{\partial \phi}{\partial \beta} &\equiv h\alpha + b\beta + f\gamma + v = 0 \\
\tfrac{1}{2}\frac{\partial \phi}{\partial \gamma} &\equiv g\alpha + f\beta + c\gamma + w = 0
\end{aligned}\right\} \quad I$$

Solving these for α, β, γ we get

$$\alpha = -\frac{Au + Hv + Gw}{D} = \frac{U}{D}$$

$$\beta = -\frac{Hu + Bv + Fw}{D} = \frac{V}{D}$$

$$\gamma = -\frac{Gu + Fv + Cw}{D} = \frac{W}{D}$$

where U, V, W are the minors of u, v, w in the determinant

$$S \equiv \begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix}$$

& $D \equiv \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

Now, if $D \neq 0$ we have a unique centre.

The constant $d' = \phi(\alpha, \beta\gamma)$

If we just multiply the left hand sides of I by $\alpha\ \beta\ \gamma$ respectively we get after addition

$$\phi(\alpha, \beta\gamma) \equiv u\alpha + v\beta + w\gamma + d.$$

$$= \frac{uU + vV + wW + dD}{D}$$

$$= \frac{S}{D}.$$

$\therefore$ The equation to the surface referred to the centre as origin is

$$U_2 + \frac{S}{D} = 0$$

Obviously if $S = 0$ the conicoid reduces to $U_2 = 0$, or a cone.

if further $D = 0$, the cone reduces to a pair of planes for $D = 0$ is the condition of U_2 being factorizable.

We shall see that by rotation of axes the form can be reduced to

$$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 + \frac{S}{D} = 0$$

Now if $D = 0$ but u, v, w not <u>all</u> zero. then in this case, there is still one definite centre, but it is at an infinite distance.

The three linear equations are parallel to one line & therefore the centre is at ∞.

This occurs when as we shall see by another Analysis.

$$\lambda_2 y^2 + \lambda_3 z^2 + 2mx = 0 \quad \text{is the form}$$

This gives two forms: —

(i) Elliptic Paraboloid when λ_2, λ_3 are of the same sign.

(ii) Hyperbolic Paraboloid when λ_2, λ_3 are of different signs.

III Suppose $D = 0$, $u = 0$, $V = 0$, $W = 0$

Then the equations of α, β, γ

become indeterminate. For the three equations are then equivalent to two

In this case there is a line centre but that line is at a finite or infinite distance.

(i) When the line is at a finite distance, the equations can be reduced to the form

$$\lambda_2 y^2 + \lambda_3 z^2 + \delta = 0 \qquad (\delta \text{ may be zero}).$$

This represents elliptic or hyperbolic cylinder or two intersecting planes (when in the last case $d = 0$).

(ii) The line will be at an infinite distance if all the three planes are parallel.

The highest terms vig u_2 form a complete square & then the equation can be thrown in the form

$$\lambda_3 z^2 + nx = 0$$

which is obviously a parabolic cylinder.

IV The three equations for planes for the centre may reduce to one. Then there is a plane centre. Then the equation must represent two parallel or two coincident planes.

A similar interpretation can be made from the tangential equation

If the determinant of the four rows & columns of the tangential equation

$$\begin{vmatrix} A & H & G & U \\ H & B & F & V \\ G & F & C & W \\ U & V & W & D \end{vmatrix} \neq 0$$

The tangential equation may represent a conicoid.

But if this vanishes we see that there are values of l, m, n p which can be determined to satisfy the four equations And this must be such that it passes through the pole of any plane whatever

The tangential equation reduces to a Conic which may degrade to two points separate or coincident.

as in the case when

$$(l\alpha + m\beta + n\gamma + p)(l\alpha' + m\beta' + n\gamma' + p) = 0$$

Sphere

The equation to the sphere can be written in either of the forms:-

$$S \equiv (x-a)^2 + (y-b)^2 + (z-c)^2 - r^2 = 0$$

$$\equiv x^2 + y^2 + z^2 + 2Ax + 2By + 2Cz + D = 0$$

In either case when the coefficient of x^2, y^2, z^2 is unity. The left hand side represents the power of the point with respect to the sphere. If the point lies on the surface, the Power is zero. — and hence the equation with right hand side put = zero.

The tangential equation, evidently is $(la + mb + nc - p)^2 = r^2(l^2 + m^2 + n^2)$

where $lx + my + nz - p = 0$ is the varying plane to touch.

If we write down equations two spheres

$$S \equiv \Sigma x^2 + 2\Sigma Ax + D = 0$$

$$S' \equiv \Sigma x^2 + 2\Sigma A'x + D' = 0$$

Introducing a linear quantity t to make it homogeneous we get

$$t\{2\Sigma(A-A')x + t(D-D')\} = 0$$

i.e. the common section of any two spheres is

$$t = 0$$

and $2\Sigma(A-A')x + t(D-D') =$ $\quad t=1$.

The first of these $t=0$ represents the fact that two spheres (any) intersect at infinity in imaginary circular section.

Intersection of two spheres is a particular case of intersection of two conicoids, which must intersect in quartic curve which however may degrade into into two plane curves.

In the more general case

$$u = 0$$
$$\text{\&}\quad u + \alpha\beta = 0$$

represent two conicoids ~~are~~ such that they have a double contact.

For the section of $\alpha = 0$, or $\beta = 0$ is the same in two conicoids.

These plane sections cut one another in two points — At these the plane section has tangents which touch both

conicoids.

$\therefore$ The conicoids have the same tangent planes.

$\therefore$ Any two spheres have a double contact at infinity.

The case of concentric spheres gives a different type of contact altogether.

$$S \equiv 0 \quad \& \quad S + \lambda = 0$$

introducing t to make it homogeneous & subtracting as before we get

$$t^2 = 0$$

The common section is repeated twice. The two spheres have a ring contact touching every point at infinity.

Example:— Prove that if two spheres have centres (a, b, c) & (a', b', c'). The equation to the two imaginary common tangent planes, ~~is~~ passing through the line of centres is:—

$$\Sigma \{(y-b)(z-c') - (y-b')(z-c)\}^2 \cancel{=0} \sqrt{(r'a - ra)} = 0$$

[Use tangential equation]. (?)

Cone

The equation of a cone of <u>any</u> degree is a homogeneous equation in the quantities

$$(x-a) \quad (y-b) \quad (z-c)$$

where (a,b,c) are the co-ordinates of the vertex.

I General problem:—

If the vertex is (a,b,c) & the generator of the cone moves through the curve defined by

$$f(x,y,z)=0$$

$$\& \quad \phi(x,y,z)=0$$

The equation to the cone can be found thus.

Any point on the generator is

$$\left(\frac{a+\lambda x}{1+\lambda}, \frac{b+\lambda y}{1+\lambda}, \frac{c+\lambda z}{1+\lambda}\right)$$

where x, y, z are the current co-ordinates of the cone.

This point must, for some value of λ must satisfy the eqs. to the curve

$$\therefore f\left\{\frac{a+\lambda x}{1+\lambda}, \frac{b+\lambda y}{1+\lambda}, \frac{c+\lambda z}{1+\lambda}\right\} = 0$$

$$\& \quad \phi\left\{\frac{a+\lambda x}{1+\lambda}, \frac{b+\lambda y}{1+\lambda}, \frac{c+\lambda z}{1+\lambda}\right\} = 0$$

If we & eliminate λ

we get the equation to the cone.

To illustrate, I

let the curve be defined

by $\quad x^2/a^2 + y^2/b^2 = 1; \quad z = 0$

& (f, g, h) the vertex.

Then we must have

$$\cancel{\frac{(f+\lambda x)^2}{a^2} + \frac{(g+\lambda y)^2}{b^2} = (1+\lambda)^2}$$

$$\cancel{\& \quad z = \frac{h+\lambda z}{1+\lambda} = 0}$$

Substituting the ~~value of~~ $\lambda = \cancel{-\frac{h}{z}}$

$$\frac{(f+\lambda x)^2}{a^2} + \frac{(g+\lambda y)^2}{b^2} = (1+\lambda)^2$$

$$\& \quad \frac{h+\lambda z}{1+\lambda} = 0$$

$$\therefore \lambda = -\frac{h}{z}$$

& we get at once

$$\frac{(hx-fz)^2}{a^2} + \frac{(hy-gz)^2}{b^2} = (z-h)^2$$

Q.E.D.

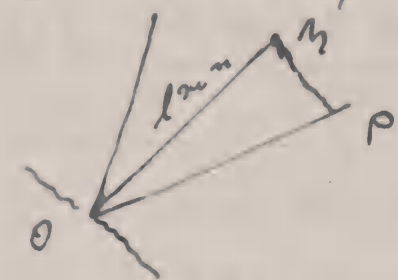
II General Problem.

To determine the conditions that the cone may be rt circular cone — i.e. a cone of revolution.

If referred to its vertex, the equation of a cone of the 2nd degree is

$$\phi(x,y,z) = ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$$

We shall compare it with the equation of a cone of revolution.



Let (l, m, n) be the direction cosines of the axis of revolution, & α the semivertical angle.

Then $OM^2 = OP^2\cos^2\alpha$

i.e. $(x^2+y^2+z^2)\cos^2\alpha = (lx+my+nz)^2$

Now if the two equations be

identical, then subtracting from (i) $\lambda \sec^2\alpha$ times (ii) we get

$$(a-\lambda)x^2 + (b-\lambda)y^2 + (c-\lambda)z^2 + 2fyz + 2gzx + 2hxy$$

$$= \text{a perfect square.}$$

$$\equiv (px + qy + rz)^2, \text{ say}$$

Then

$$\begin{array}{l|l} (a-\lambda) = p^2 & f = qr \\ b-\lambda = q^2 & g = rp \\ c-\lambda = r^2 & h = pq. \end{array}$$

If none of the quantities f, g, h be zero, then p, q, r, none can be zero. & eliminating we get

$$\lambda = a - \frac{gh}{f} = b - \frac{hf}{g} = c - \frac{fg}{h}.$$

These are the two conditions of the cone being one of revolution.

(ii) Suppose now <u>one</u> of the quantities say $f = 0$. No circular cone is possible because if $f = 0$, q, or $r = 0$

& $\therefore$ h or $g = 0$

Thus at least two must vanish, if one is to vanish at all.

(iii) Suppose, then

$$f = 0, \; g = 0 \quad \text{but } h \neq 0$$

Then $\nu = 0$ & $\lambda = c$

And eliminating $p \cdot q$ between (i) (ii) & (vi) we get

$$(a-c)(b-c) = h^2.$$

(iv) If, however, $f = g = h = 0$

Then no elimination is necessary & the cone of revolution is obtained.

III The most important point about cones shall be considered now.

Let by any transformation of rectangular axes, origin remaining unchanged

The expression

$$u_2 - \lambda(x^2+y^2+z^2) = 0$$

is changed into $u'_2 - \lambda(x'^2+y'^2+z'^2) = 0$

Then if the 1st represents two planes, 2nd will also do.

The conditions of λ must be the same.

i.e.
$$\begin{vmatrix} a-\lambda & g & h \\ g & b-\lambda & f \\ h & f & c-\lambda \end{vmatrix} = 0$$

&
$$\begin{vmatrix} a'-\lambda & g' & h' \\ g' & b'-\lambda & f' \\ h' & f' & c'-\lambda \end{vmatrix} = 0$$

must lead to the same values of λ.

Comparing the coefficients we get

$$a+b+c = a'+b'+c'. \quad \cdots (1)$$

$$\Sigma bc - \Sigma f^2 = \Sigma b'c' - \Sigma f'^2 \quad \cdots (2)$$

& $$D = D'. \quad \cdots (3)$$

These three are then the invariants.

IV To interprete what happens in the case of a cone if any of these invariants vanishes.

Lect: on 25th October 1921. 27th Oct. 1921.

The problem of finding the cylinder whose generators are parallel to

$$x/l = y/m = z/n$$

& for which a curve traced out is known to be

$$f(x, y, z) = 0$$
$$\phi(x, y, z) = 0$$

is solved in the same way as that of the cone.

Let x, y, z be a point on the surface: then $(x + lr, y + mr, z + nr)$ is any point on the generator & it must lie on the curve for some value of r.

$$\therefore \quad f(lr + x, y + mr, z + nr) = 0$$

$$\phi(x + lr, y + mr, z + nr) = 0$$

The elimination of r between these gives the equation to the cylinder

If however, instead of giving a curve it is given that the generators touch a certain surface say

$$F(x, y, z) = 0$$

then we can proceed just this way.

$$F(lr + x, mr + y, nr + z) = 0$$

And the condition of tangency is that two points should coincide; and hence the algebraic condition of there being two equal roots for r gives the equation to the cylinder.

"Central Conicoids".

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \text{-- Ellipsoid}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \quad \text{-- Hyperboloid of one sheet}$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \quad \text{-- " of two sheets}$$

Properties of normals:-

We shall confine our attention only to the case of an ellipsoid. Other cases being obtained by simply changing the signs of b^2 or c^2.

The normal at $p(x_1, y_1, z_1)$ is

$$\frac{x - x_1}{x_1/a^2} = \frac{y - y_1}{y_1/b^2} = \frac{z - z_1}{z_1/c^2} = pr.$$

where p = perpendicular from the origin on the tangent plane at p.

$$= 1 / + \sqrt{\Sigma \; x_1^2/a^4}$$

In taking the positive sign for the radical we measure r positively when drawn outwards from the ellipsoid.

And any point on the ellipsoid-normal is thus

$$\left\{ \left(1 + \frac{rp}{a^2}\right)x_1, \left(1 + \frac{rp}{b^2}\right)y_1, \left(1 + \frac{rp}{c^2}\right)z_1 \right\}.$$

Now the lengths of the intercepts on the normals between the surface &

and the co-ordinate planes are

$$-\frac{a^2}{p}, \ -\frac{b^2}{p}, \ -\frac{c^2}{p}. \quad \text{respectively.}$$

$$\left.\begin{aligned} PG_1 &= -\frac{a^2}{p} \\ PG_2 &= -b^2/p \\ PG_3 &= -c^2/p \end{aligned}\right\}$$ The normal at P meets the planes $x=0$ $y=0$, $z=0$ in G_1 G_2 G_3 respectively.

$\therefore$ We easily get,

$$G_2G_3 : G_3G_1 : G_1G_2 = (b^2-c^2) : (c^2-a^2) : (a^2-b^2).$$

Thus interpreted in words every normal possesses the property that its intercepts between the co-ordinate planes, taken two by two are in a constant ratio.

The converse is not true; For, this is only one condition. In order to determine whether a given st.-line is normal to the surface or not, two conditions are necessary — since

there is a double infinity of normals.

Really speaking the condition that we have obtained is that "the ~~normal~~ line should be perpendicular to its conjugate line" – of which the condition is

$$\Sigma a^2 l \lambda = 0$$

We proceed to prove that

$$\frac{G_2 G_3}{b^2-c^2} = \frac{G_3 G_1}{c^2-a^2} = \frac{G_1 G_2}{a^2-b^2}$$

&

$$\Sigma a^2 l \lambda = 0$$

are really identical.

For If $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ be ~~the~~ a line meeting the co-ordinates planes in $G_1 G_2 G_3$, – then

$$PG_1 = -\frac{\alpha}{l} \quad \&c.$$

$$\therefore G_2 G_3 = \frac{\gamma}{n} - \frac{\beta}{m} ; \&c.$$

$$= \frac{m\gamma - n\beta}{mn} = \frac{\lambda}{mn}$$

$$\therefore \quad \frac{\lambda}{mn(b^2-c^2)} = \frac{\mu}{nl(c^2-a^2)} = \frac{\nu}{lm(a^2-b^2)}$$

$$\therefore \quad \frac{a^2 l \lambda}{a^2(b^2-c^2)} = \frac{b^2 m \mu}{b^2(c^2-a^2)} = \frac{c^2 n \nu}{c^2(a^2-b^2)}$$

$$= \frac{\Sigma a^2 l \lambda}{0}$$

$$\therefore \quad \Sigma a^2 l \lambda = 0$$

Q.E.D.

In fact the condition

$\Sigma a^2 l \lambda = 0$ determines

that ~~whether~~ the line is normal to <u>some</u> confocal ellipsoid.

Examples for homework:—

(1.) Show that the normals from a fixed point to all the confocals lie on a cone of the second degree, which is of the type that it has three perpendicular generators.

(2) Show that the condition $\Sigma a^2 l \lambda = 0$

is satisfied by a chord of the ellipsoid if the normals at its extremities intersect.

Show also that such a chord is the axis of one plane section of the ellipsoid.

(3) Show that the locus of a point such that the sum of the squares on the six normals drawn from it to the ellipsoid is constant $= k^2$ is a concentric coaxal conic of the form,

$$Af^2 + Bg^2 + Ch^2 = \text{constant}.$$

Lecture 27th Oct. 1921 27th Oct. 1921.

"Conjugate diameters"

On the ellipsoid $\Sigma x^2/a^2 = 1$
we can take three points $(x_r y_r z_r)$
$r = 1, 2, 3$

such that $\Sigma \frac{x_r x_s}{a^2} = 0$

To treat, however, the theory in a rather different way:—

Transform the ellipsoid into the corresponding sphere by the following substitutions:—

$$\xi = \frac{x}{a}\,;\ \eta = \frac{y}{b}\,;\ \zeta = \frac{z}{c}$$

Then $\xi^2 + \eta^2 + \zeta^2 = 1$

Now the projection leaves all properties of ratios of lengths unaltered in going from the sphere to the ellipsoid & so on.

In the case of a sphere conjugate diameters are mutually perpendicular.

$\therefore$ If we choose three diameters so as to correspond to a left handed system, in a sphere — we shall preferably take its to be of unit radius, we must have all the relation of three mutually perpendicular lines.

(i)
$$\begin{vmatrix} \xi_1 & \eta_1 & \zeta_1 \\ \xi_2 & \eta_2 & \zeta_2 \\ \xi_3 & \eta_3 & \zeta_3 \end{vmatrix} = +1$$

(ii) each element = its minor

(iii) Sum of the squares of the elements of any row or any column = 1

(iv) & Sum of the product of the corresponding elements of any two rows or columns = 0

Hence we get,

$$\begin{vmatrix} \frac{x_1}{a}, & \frac{y_1}{b}, & \frac{z_1}{c} \\ \frac{x_2}{a}, & \frac{y_2}{b}, & \frac{z_2}{c} \\ \frac{x_3}{a}, & \frac{y_3}{b}, & \frac{z_3}{c} \end{vmatrix}$$

· a determinant which the same properties.

Thus

(1) $\Delta = +1$

$$\therefore \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = abc.$$

But the left hand side is the six times the volume of the tetrahedron whose vertex is at 0, the centre, & base the triangle $(x_r y_r z_r)$ $r = 1, 2, 3$.

= 2 the volume of the circumscribing ▭ piped

∴ Volume of the ellips ▭ ped circumscribing the ellipsoid $= 8abc$.

(2) $\Sigma x_1^2 = a^2$; $\Sigma y_1^2 = b^2$; $\Sigma z_1^2 = c^2$.

$\therefore r_1^2 + r_2^2 + r_3^2 = a^2 + b^2 + c^2$.

(3) $\Sigma \frac{x_1 x_2}{a^2} + \frac{y_1 y_2}{b^2} + \frac{z_1 z_2}{c^2} = 0$

&c.

P.T.O.

(4) $$\frac{1}{p_r^2} = \frac{x_r^2}{a^4} + \frac{y_r^2}{b^4} + \frac{z_r^2}{c^4}$$

$$\therefore \frac{1}{p_1^2} + \frac{1}{p_2^2} + \frac{1}{p_3^2} = \left[\frac{\Sigma x_r^2}{a^2} + \frac{\Sigma y_r^2}{b^2} + \frac{\Sigma z_r^2}{c^2} \right]_{\cancel{r=1}}^{\cancel{r=3}}$$

$(r = 1, 2, 3)$

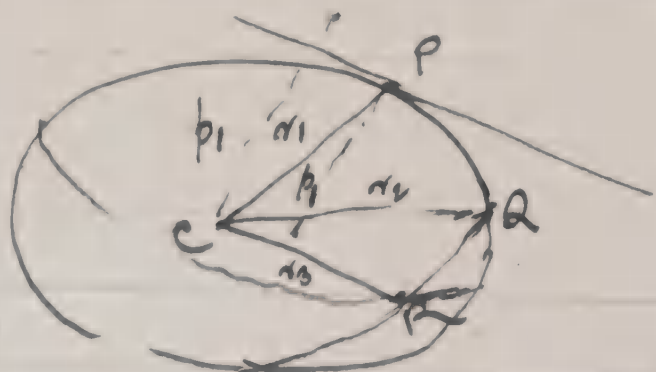
$$= \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}.$$

$$\therefore \Sigma b^2c^2 = \frac{a^2b^2c^2}{p_1^2} + \frac{a^2b^2c^2}{p_2^2} + \frac{a^2b^2c^2}{p_3^2}.$$

Now since by (1) the ▭piped is of constant volume, we get

$$\therefore p_1 r_2 r_3 \sin \widehat{r_2 r_3} = abc$$?

$$\therefore \Sigma r_2^2 r_3^2 \sin^2 \widehat{r_2, r_3} = \Sigma b^2c^2.$$

∴ 

25th Dec 192?

Area $CQR = \frac{1}{2} r_2 r_3 \sin \widehat{r_2 r_3}$

$\therefore$ Vol. of $CPQR = \frac{1}{6} p_1 r_2 r_3 \sin \widehat{r_2 r_3}$ } ∴ in vain

$= \frac{1}{6} abc$

When we take the hyperboloid of one sheet we can prove the properties of conjugate diameters by the same method. But an imaginary axis will have to be taken.

The locus of a point, the sum of the squares of whose normals from the ellipsoid $S \equiv \Sigma \frac{x^2}{a^2} - 1 = 0$, is constant $(=k^2)$ is

$\frac{Ex. 23}{p. 129}$ Bell

$$6 \Sigma x^2 - 2 \Sigma \left\{ \frac{a^2x^2(b^2+c^2-2a^2)}{(a^2-b^2)(c^2-a^2)} \right\} + 2 \Sigma a^2 = k^2.$$

Proof:– let (f, g, h) be the point.

If the normal at say (x, y, z) viz

$$\frac{X - x}{\frac{x}{a^2}} = \frac{Y - y}{y/b^2} = \frac{Z - z}{z/c^2} = \lambda$$

passes through (f, g, h),

$$\frac{f - x}{x/a^2} = \quad = \quad = \lambda .$$

$$\therefore \quad f = x\left(1 + \frac{\lambda}{a^2}\right) \quad ; \ \&c.$$

$$\therefore \quad x = \frac{a^2 f}{a^2+\lambda} \ ; \ y = \frac{b^2 g}{b^2+\lambda} \ ; \ z = \frac{c^2 h}{c^2+\lambda}.$$

But (x, y, z) lies on $S = 0$

$$\therefore \quad \Sigma \frac{a^2 f^2}{(a^2+\lambda)^2} = 1.$$

This is a sextic in λ giving six values, corresponding to which there are six normals.

Solving the equation could be written

$$\begin{aligned}\phi(\lambda) \equiv\ & (\lambda+a^2)^2(\lambda+b^2)^2(\lambda+c^2)^2 \\ & - a^2f^2(\lambda+b^2)^2(\lambda+c^2)^2 \\ & - b^2g^2(\lambda+c^2)^2(\lambda+a^2)^2 \\ & - c^2h^2(\lambda+a^2)^2(\lambda+b^2)^2\end{aligned}$$

$$\equiv (\lambda-\lambda_1)(\lambda-\lambda_2) - - (\lambda-\lambda_r) = 0.$$

It is easy to see from this that

$$\frac{\phi'(\lambda)}{\phi(\lambda)} = \sum_1^6 \frac{1}{\lambda-\lambda_r}$$

Now if $(x_r\ y_r\ z_r)$ is the foot of a normal n_r

$$n_r^2 = (x_r-f)^2+(y_r-g)^2+(z_r-h)^2$$

But we have found that

$$x_r = \frac{a^2f}{a^2+\lambda_r}$$

$$\therefore\ x_r-f = \frac{-f\lambda_r}{a^2+\lambda_r}.$$

$$\therefore\ n_r^2 = \lambda_r^2\left[\frac{f^2}{(a^2+\lambda_r)^2}+\frac{g^2}{(b^2+\lambda_r)^2}+\frac{h^2}{(c^2+\lambda_r)^2}\right].$$

Adding $f^2+g^2+h^2$ & subtracting each from each term corresponding.

we get

$$n_r^2 = f^2+g^2+h^2 - \left[\Sigma \frac{(2a^2\lambda_r + a^4) f^2}{(a^2+\lambda_r)^2}\right]$$

Now the bracketed expression could be split up into parts thus

$$\Sigma \frac{(2a^2\lambda_r + a^4) f^2}{(a^2+\lambda)^2} = \lambda_r \Sigma \frac{a^2 f^2}{(a^2+\lambda_r)^2} + \Sigma \frac{a^2 f^2}{a^2+\lambda}.$$

$$\therefore \sum_1^6 n_r^2 = 6(f^2+g^2+h^2) - \sum_1^6 \lambda_r - \sum_1^6 \left[\Sigma \frac{a^2 f^2}{a^2+\lambda}\right]$$

$$\left[\because \Sigma \frac{a^2 f^2}{(a^2+\lambda)^2} = 1 .\right.$$

It is easily seen, moreover that

$$-\Sigma \lambda_r = 2 \Sigma a^2 .$$

$$\therefore \sum_1^6 n_r^2 = 6 \Sigma f^2 + 2 \Sigma a^2 - \sum_1^6 \Sigma \frac{a^2 f^2}{a^2+\lambda}$$

Thus the problem reduces to the evaluation of $\Sigma \Sigma \frac{a^2 f^2}{a^2+\lambda}$.

$$= \sum_1^6 \frac{a^2 f^2}{a^2+\lambda_r} + \sum_1^6 \frac{b^2 g^2}{b^2+\lambda} + \sum_1^6 \frac{c^2 h^2}{c^2+\lambda}.$$

Now we know that

$$\frac{\phi'(\lambda)}{\phi(\lambda)} = \sum_1^6 \frac{1}{\lambda - \lambda_r}$$

$$\therefore -\frac{\phi'(-a^2)}{\phi(-a^2)} = \sum_1^6 \frac{1}{a^2+\lambda_r}$$

$$\therefore \sum_1^6 \frac{a^2f^2}{a^2+\lambda_r} = -\frac{\phi'(-a^2)}{\phi(-a^2)}\, a^2f^2$$

Now $\phi(-a^2) \equiv -a^2f^2(b-a^2)^2(c^2-a^2)^2$.

To find $\phi'(-a^2)$, we observe that in $\phi(\lambda)$ all the terms except one viz $-a^2f^2(\lambda+b^2)^2(\lambda+c^2)^2$ contain $(a^2+\lambda)^2$ as a factor.

$\therefore$ $\phi'(\lambda)$, all these terms will contain $(a^2+\lambda)$ as a factor.

$\therefore$ We need consider only the term $-a^2f^2(\lambda+b^2)^2(\lambda+c^2)^2$

Taking logarithmic differentiation

$$\frac{\phi'(\lambda)}{\phi(\lambda)}_{\lambda\to -a^2} = \frac{2}{\lambda+b^2} + \frac{2}{\lambda+c^2}$$

$$= 2\ \frac{1}{b^2-a^2} + \frac{1}{c^2-a^2} = \frac{2(b^2+c^2-2a^2)}{(b^2-a^2)(c^2-a^2)}$$

$$\therefore \quad \sum_1^6 \frac{a^2f^2}{a^2+\lambda} = \frac{2a^2f^2(b^2+c^2-2a^2)}{(a^2-b^2)(c^2-a^2)}.$$

Similarly with $\sum_1^6 \frac{b^2g^2}{b^2+\lambda}$, & $\sum_1^6 \frac{c^2h^2}{c^2+\lambda_r}$.

$\therefore$ The locus of (f, g, h) is obtained finally in the form

$$6\sum x^2 - 2\sum \frac{a^2x^2(b^2+c^2-2a^2)}{(a^2-b^2)(c^2-a^2)} + 2\sum a^2 = k^2$$

Q.E.D.

Lec. on 29th Oct. 1921 29th Oct. 1921.

Plane sections of a Central Conicoid.

1°. Let us suppose that the conicoid is referred to a point on it such that $z=0$ is the tangent plane at the origin. i.e. axis of z is the normal at the point.

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2wz=0$$

All parallel sections of a conicoid are similar conics.

If we put $z=k$, the section determined is the same as its projn. on xy plane & is therefore

$$ax^2+by^2+2hxy+2fyk+2gkx+ck^2+2wk=0$$

And these conics are all similar since they have the same terms of the second degree.

The statement is also true of similar & similarly situated conicoids. For the terms of the highest degree (2nd) are the same in all.

One point more. If we take the

Section by parallel tangent plane the section is always two lines.

For putting $z=0$, which is the tangent plane we get

$$ax^2+2hxy+by^2=0$$

which in all cases represent two st. lines, real, imaginary or coincident.

This last fact enables us to determine whether a given section is an ellipse, hyperbola &c.

To take an illustrative example,

Determine the nature of the section

of $\quad x^2/a^2 + y^2/b^2 - z^2/c^2 = 1$

by $\quad lx+my+nz-p=0$

Now since parallel sections of similar & similarly situated conicoids are similar conics.

$\therefore$ It is sufficient to examine the nature of the section of

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$$

by $\quad lx+my+nz=0$

eliminating z, we get

$$n^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right) = \frac{(lx+my)^2}{c^2}$$

i.e. $$x^2\left(\frac{n^2}{a^2}-\frac{l^2}{c^2}\right) - \frac{2lmxy}{c^2} + y^2\left(\frac{n^2}{b^2}-\frac{m^2}{c^2}\right) = 0$$

These are two st. lines, & are real & distinct, real & ~~coinc~~ coincident or imaginary acc. as

$$\frac{l^2m^2}{c^4} > = \text{ or } < \left(\frac{n^2}{a^2}-\frac{l^2}{c^2}\right)\left(\frac{n^2}{b^2}-\frac{m^2}{c^2}\right).$$

i.e. acc. as $0 > = \text{ or } < \frac{n^4}{a^2b^2} - \frac{l^2n^2}{b^2c^2} - \frac{m^2n^2}{a^2c^2}$

i.e. according as

$$a^2l^2+b^2m^2 \substack{>\\=\\<} c^2n^2$$

It follows from this that the original section also,

of $$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2} = 1$$

by $$lx+my+nz = p$$

? is an ellipse, parabola, or hyperbola

notation wrong acc. as $$a^2l^2+b^2m^2 \substack{>\\=\\<} c^2n^2$$

What is to be noted however in these last

two cases is this:-

The parabolas may degrade into two parallel st. lines if $p = 0$

For with $p = 0$ & the condition for parabola viz $a^2l^2 + b^2m^2 = c^2n^2$, the plane is found to touch the cone which is asymptotic.

The hyperbola, in the last case may degrade into two st lines, if in particular $a^2l^2 + b^2m^2 - c^2n^2 = p^2$.

2°. To find the lengths of Axes of plane sections: —

Let the Conicoid be

$$S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$$

& the cutting plane

$$lx + my + nz = 0$$

Consider those radii of the ellipsoid which are of length r.

They satisfy the equation

$$\frac{x^2}{a^2} \quad x^2 + y^2 + z^2 = r^2$$

$$\therefore \quad x^2\left(\frac{1}{a^2} - \frac{1}{r^2}\right) + y^2\left(\frac{1}{b^2} - \frac{1}{r^2}\right) + z^2\left(\frac{1}{c^2} - \frac{1}{r^2}\right) = 0$$

will contain these.

The plane $lx + my + nz = 0$ will cut this cone in two lines of length r, which will coincide if & only if r is one of the semi-axes.

$\therefore$ Therefore the condition of contact

viz.

$$\Sigma \frac{l^2}{\frac{1}{a^2} - \frac{1}{r^2}} = 0$$

gives the values of r.

i.e. $\Sigma \dfrac{a^2l^2}{a^2-r^2} = 0$

i.e. $\Sigma(b^2-r^2)(c^2-r^2)a^2l^2 = 0$

i.e. $r^4\Sigma a^2l^2 - r^2\Sigma a^2(b^2+c^2)l^2 + \Sigma b^2c^2a^2l^2 = 0$

$\therefore$ If r_1 & r_2 be the semi axes.

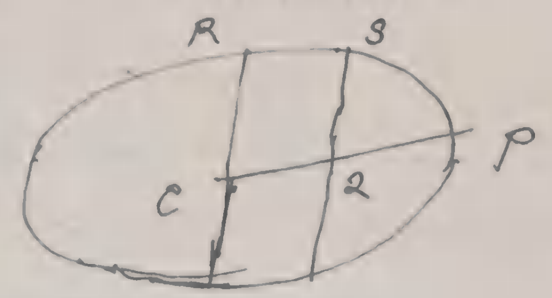
$$r_1^2 + r_2^2 = \frac{\Sigma a^2l^2(b^2+c^2)}{\Sigma a^2l^2}$$

& $$r_1^2 \cdot r_2^2 = \frac{a^2b^2c^2\,\Sigma l^2}{\Sigma a^2l^2}.$$

Let us now obtain from this the axes of the section by the plane 3°.

$$lx + my + nz = p.$$

Geometrical Solution:–



Let QS be an axis of the plane section
$lx + my + nz - p = 0$
& CR that of the section by $lx+my+nz=0$

CQP being a diameter of the ellipsoid.

Then it is a well-known property of an ellipse that

$$QS^2 = CR^2\left(1 - \frac{CQ^2}{CP^2}\right)$$

Eq. referred to conjugate axes is
$x^2/a'^2 + y^2/b'^2 = 1$
i.e. $CQ^2/CP^2 + SQ^2/CR^2 = 1$

But $\dfrac{CQ}{CP}$ = ratio of the perpendiculars from C on the planes through ~~R & Q~~ Q & P

$$= p \Big/ \sqrt{\Sigma a^2 l^2}$$

$$\therefore 1 - \frac{CQ^2}{CP^2} = 1 - \frac{p^2}{p_0^2} = 1 - \frac{p^2}{\Sigma a^2 l^2}$$

$\therefore$ If R_1 R_2 be the semi-axes in this section, we easily get from the results in 2° that

$$R_1^2 + R_2^2 = (r_1^2 + r_2^2)\left(1 - \frac{p^2}{p_0^2}\right)$$

$$= \left(1 - \frac{p^2}{p_0^2}\right) \cdot \frac{\Sigma a^2 l^2 (b^2 + c^2)}{\Sigma a^2 l^2}$$

& $$R_1^2 \cdot R_2^2 = r_1^2 \cdot r_2^2 \left(1 - \frac{p^2}{p_0^2}\right)^2$$

$$= \left(1 - \frac{p^2}{p_0^2}\right)^2 \cdot \frac{a^2 b^2 c^2 \Sigma l^2}{\Sigma a^2 l^2}$$

Q.E.D.

P.T.O.

These results are equally valid for hyperboloids of one sheet or two sheets. The necessary changes of signs being made.

In some cases the results require interpretation. Because if the sections are hyperbolas, one of the two quantities r_1^2 & r_2^2 would be negative.

If the section is a parabola, to find the latus rectum of the parabola is a problem. 4°

In this case $\Sigma a^2 l^2 = 0$

Though in this case, thus, both

? × ~~r_1 & $r_2 = \infty$~~ × (constant) still, we have the relation

$r_1 . r_2 = \infty$ ~~2AM~~

$r_2^2 = r_1 L$. where L is the semi-latus rectum of the parabola:

Lect. on 1st Nov. 1921 1 Nov. 1921

If we change $\frac{1}{a^2}$ into a, $\frac{1}{b^2}$ into b &c. in the formulae obtained last time we get, the axes of the section of

$$ax^2 + by^2 + cz^2 = 1$$

by $lx + my + nz - p = 0$

& we have

$$r_1^2 + r_2^2 = + \frac{\Sigma \frac{l^2}{a}\left(\frac{1}{b} + \frac{1}{c}\right)}{\Sigma \frac{l^2}{a}} \left(1 - \frac{p^2}{\Sigma \frac{l^2}{a}}\right).$$

$$\& \quad r_1^2 \cdot r_2^2 = \frac{\frac{1}{abc} \Sigma l^2}{\Sigma \frac{l^2}{a}} \left(1 - \frac{p^2}{\Sigma \frac{l^2}{a}}\right)^2$$

Now if the section be a parabola which happens when $\Sigma \frac{l^2}{a} = 0$,

we have

$$r_1^2 + r_2^2 = r_1^2 + L r_1 \quad [\sim r_1^2 \text{ ultimately}]$$

$$\& \quad r_1^2 \cdot r_2^2 = L r_1^3$$

$$\therefore 1/L = \text{Lt.} \frac{(r_1^2 + r_2^2)^{3/2}}{r_1^2 r_2^2}.$$

$$= \cancel{\#} \cdot \frac{\left\{\Sigma \frac{l^2}{a}\left(\frac{1}{b} + \frac{1}{c}\right)\right\}^{3/2}}{\frac{1}{abc}\Sigma l^2} \cdot \frac{p^3}{p^4}. \quad ?$$

Taking limit by putting $\Sigma l^2/a = 0$

Now $\frac{l^2}{a}\left(\frac{1}{b}+\frac{1}{c}\right)$

$$= \left(\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}\right)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) - \left(\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}\right)$$

$$= -\Sigma \frac{l^2}{a^2}$$

$\therefore$ The latus rectum is given by

$$L \cdot \left\{\Sigma \frac{l^2}{a^2}\right\}^{\frac{3}{2}} = \frac{p}{abc} \Sigma l^2.$$

The foregoing equation ought to be written with $L, \pm$.

If we take the plane such that p is +ve, & the surface be a hyperboloid of one sheet, abc is −ve & $\therefore$ we must take negative sign with L.

If, on the other hand, the surface be a hyperboloid of two sheets, abc is +ve, & we must take +ve sign with L.

To determine the axes of the section of

$$\frac{y^2}{L} + \frac{z^2}{L'} = 2x$$

by

$$lx + my + nz - p = 0$$

The axes are the same as those of the section of

$$\frac{y^2}{L} + \frac{z^2}{L'} = 2\left(\frac{p}{l} - \frac{my + nz}{l}\right).$$

by $lx + my + nz - p = 0$

Now this is a cylinder with its axis parallel to Ox.

The equation can be written in the form

$$\left\{\frac{y^2}{L} + \frac{2my}{l} + \frac{m^2}{l^2}\cdot L\right\} + \left\{\frac{z^2}{L'} + \frac{2nz}{l} + \frac{n^2}{l^2}L'\right\}$$

$$= \frac{2p}{l} + \frac{m^2}{l^2}L + \frac{n^2}{l^2}L'$$

i.e. $\left\{\frac{y}{\sqrt{L}} + \frac{m}{l}\sqrt{L}\right\}^2 + \left\{\frac{z}{\sqrt{L'}} + \frac{n}{l}\sqrt{L'}\right\}^2$

$$\equiv \lambda \text{, say.}$$

This cylinder is to be cut by the plane

$$lx + my + nz - p = 0$$

But parallel sections of the cylinder are all the same

Therefore we may merely take the cylinder transferred to a point on its axis, as origin, viz.

$$\frac{y^2}{\lambda L} + \frac{z^2}{\lambda L'} = 1$$

cut by the plane $lx + my + nz = 0$

But this is exactly the same problem as that of the ellipsoid, where the axis along Ox is made ∞.
i.e. $a^2 \to \infty$.

The quadratic in r^2 becomes

$$l^2 + \frac{\lambda L m^2}{\lambda L - r^2} + \frac{\lambda L' n^2}{\lambda L' - r^2} = 0$$

which reduces to the form,

$$l^2 r^4 - r^2 \lambda \left\{ (l^2 + m^2) L + (l^2 + n^2) L' \right\}$$
$$+ \lambda^2 L L' (l^2 + m^2 + n^2) = 0$$

From which we get,

$$r_1^2 + r_2^2 = \frac{L(l^2+m^2) + L'(l^2+n^2)}{l^2}\lambda$$

$$= \frac{L(l^2+m^2) + L'(l^2+n^2)}{l^4}(2pl + Lm^2 + L'n^2).$$

Similly $$r_1^2 \cdot r_2^2 = \frac{LL'(l^2+m^2+n^2)}{l^6}(2pl + Lm^2 + L'n^2)^2.$$

The Case when $l = 0$,

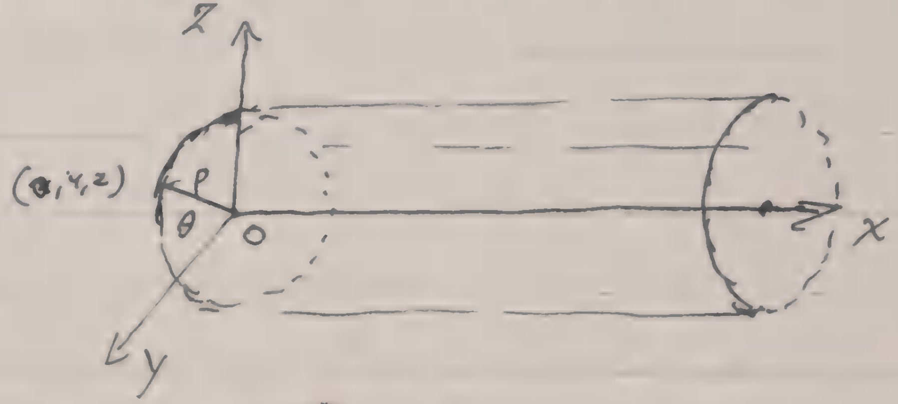
i.e. when the plane $lx + my + nz - p = 0$ is parallel to the axis of the paraboloid, the section will be parabola.

In order to find the latus rectum of the parabola we use the fact that all sections parallel to the axes & parallel to one another are equal parabolas, — of the same latus rectum.

$\therefore$ We have merely to take a parabolic section through the axis of x.

P.T.O.

At this stage it is best to use cylindrical Co-ordinates:—



$$y = \rho \cos\theta$$
$$z = \rho \sin\theta.$$

& we get
$$\frac{y^2}{L} + \frac{z^2}{L'} = 2x$$
transformed into—

$$\rho^2\left(\frac{\cos^2\theta}{L} + \frac{\sin^2\theta}{L'}\right) = 2x.$$

Now if the section be taken at an angle θ with the xy plane the the equation to the parabola of section is
$$\rho^2 = 2\left(\frac{\cos^2\theta}{L} + \frac{\sin^2\theta}{L'}\right)^{-1} x$$

Comparing this with
$$y^2 = 2lx$$
we get

~~The Semi latus~~

The Latus Rectum $= \dfrac{\cos^2\theta}{L} + \dfrac{\sin^2\theta}{L'}$.

Q.E.D.

———— o o o o ————

"Circular Sections"

If in our previous investigations we make $r_1^2 = r_2^2$ the section in question will become a ~~sph~~ circle. ~~In~~ For that, we shall get a condition of the form

$(a-b)^2 + 4h^2 = 0$ which is equivalent to two conditions $a = b$ & $h = 0$

The following is a better way:-

If $S = 0$ has one circular section, it has an infinity of them; for, parallel sections are similar. Let us attempt to find that one which passes through the origin the centre of $S=0$

If r is the radius of the circle it will also lie upon the sphere

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2} = 1$$

Now the cone through this curve with vertex at the origin is

$$\Sigma\, x^2\left(\frac{1}{a^2} - \frac{1}{r^2}\right) = 0$$

The cone must degrade into two planes

i.e. $\Pi\left(\frac{1}{a^2}-\frac{1}{r^2}\right) = 0$

i.e. $r^2 = a^2, b^2,$ or c^2.

when $a^2 > b^2 > c^2$

the value $r^2 = b^2$ alone gives a real plane

And the circular sections of the ellipsoids are given by the planes

$$x^2\left(\frac{1}{b^2}-\frac{1}{a^2}\right) = z^2\left(\frac{1}{c^2}-\frac{1}{b^2}\right).$$

And these are equally inclined to the axes.

The form is $p^2x^2 - q^2y^2 = 0$

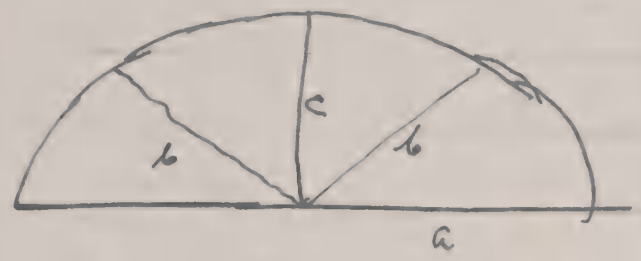
& hence

$$\left.\begin{aligned} px + qy + \alpha &= 0 \\ \& \quad px - qy + \beta &= 0 \end{aligned}\right\}$$

form two systems of planes which give circular sections.

"Geometrical Consideration"

If b is intermediate between c & a



it is easy to see that it is possible to draw two radii each equal to b in the elliptic section.

Otherwise, a, c being the max. & min. radii vectors no radius whose length does not lie between these is possible.

Hence $r^2 = (\text{Intermediate semi axis})^2$ gives the real solution.

Just in the same way the problem of the circular sections of the two other conicoids can be treated.

P. T. O.

Lect. on 3rd Nov. 1921 3rd Nov. 1921

We have already seen that the tangent plane of a conicoid at any pt. cuts it in two real, or imaginary lines. It is evident that the st. lines will be real only in the case when the surface of the conicoid bends in two opposite directions. Such surfaces are called Anticlastic.

The only two cases of a real conicoid of this nature are the hyperboloid of one sheet, & the hyperbolic parabola.

There are of course st. lines on the cone & the cylinder, where of course the tangent plane meets the surface in two coincident lines.

To find the generators of

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

Now the principal Elliptic section is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ & $z = 0$

And $(a\cos\alpha, b\sin\alpha, 0)$ is any point on the

The tangent plane at this point to the surface is

$$\frac{x a \cos\alpha}{a^2} + \frac{y b \sin\alpha}{b^2} = 1$$

i.e. $$\frac{x \cos\alpha}{a} + \frac{y \sin\alpha}{b} = \cos^2\alpha + \sin^2\alpha$$

$$\therefore \quad \frac{\cos\alpha\,(x - a\cos\alpha)}{a} = -\frac{\sin\alpha\,(y - b\sin\alpha)}{b}$$

$$\therefore \quad \frac{x - a\cos\alpha}{a\sin\alpha} = \frac{y - b\sin\alpha}{-b\cos\alpha}.$$

If this plane cuts the surface in two lines, the equations to the line must of the form

$$\frac{x - a\cos\alpha}{a\sin\alpha} = \frac{y - b\sin\alpha}{-b\cos\alpha} = \frac{z}{k}.$$

which gives

$$\frac{x}{a} = \cos\alpha + \sin\alpha\,\frac{z}{k}$$

$$\frac{y}{b} = \sin\alpha - \cos\alpha\,\frac{z}{k}.$$

Substituting these values in the equation of the surface,

$$1 + \frac{z^2}{k^2} = 1 + \frac{z^2}{c^2}$$

$\therefore k = \pm c$, if the lines are to lie on the surface.

We have thus found two sets of generators

$$\frac{x - a\cos\alpha}{a\sin\alpha} = \frac{y - b\sin\alpha}{-b\cos\alpha} = +\frac{z}{c}$$

$$\frac{x - a\cos\alpha}{a\sin\alpha} = \frac{y - b\sin\alpha}{-b\cos\alpha} = -\frac{z}{c}.$$

The first set pass in the direction of z increasing with x $\nearrow$.

The second pass in the direction of z decreasing with x $\longrightarrow$.

It is clear that two generators of the same system cannot intersect while two generators of opposite systems do intersect.

Let us take now two generators of two systems each (α, β points)

They lie in the tangent planes at α, β; which as we have seen are vertical.

$\therefore$ Their points of intersection lies vertically over the intersection of the tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the points α, β.

$\therefore$ The co-ordinates of the point where the generators meet are

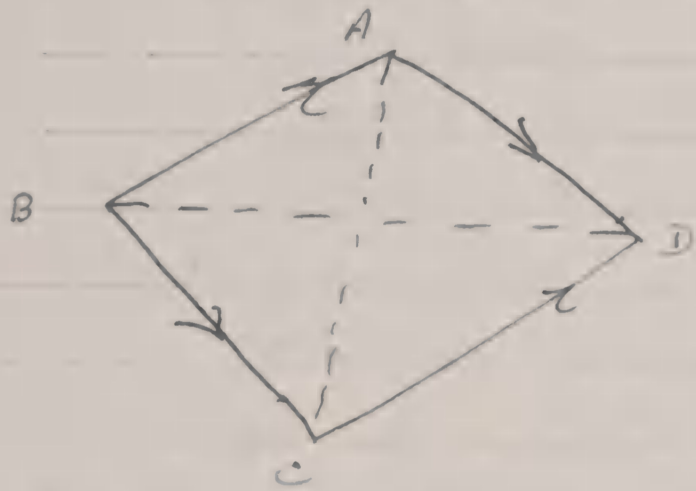
$$T\left(a\,\frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\beta-\alpha}{2}},\ b\,\frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\beta-\alpha}{2}},\ c\tan\frac{\alpha-\beta}{2}\right).$$

$\therefore$ We see that any point on the hyperboloid is parametrically represented by the co-ordinates

$$x = a\cos\theta\sec\phi$$
$$y = b\sin\theta\sec\phi$$
$$z = c\tan\phi.$$

Let us take a Conicoid & draw on it four generators, two of each system.

We shall then get a skew quadrilateral



AB, CD belong to one system

BC, AD ,, to the second system.

AC, BD do not lie on the Surface but they are conjugate lines [For Proof:—

ACD is the tangent plane at D

ABC ,, ,, ,, ,, B

$\therefore$ The tangent planes at B, D intersect in AC

$\therefore$ AC, BD are conjugate].

Now ABCD is a tetrahedron, & if we take them as our planes of reference, the equations to the plane being

$\alpha = 0$; $\beta = 0$; $\gamma = 0$; & $\delta = 0$

The equations to the lines are easily written down :—

$$AB \longrightarrow \delta = \gamma = 0$$
$$BC \longrightarrow \alpha = \beta = 0$$
$$CD \longrightarrow \beta = \alpha = 0$$
& $$DA \longrightarrow \beta = \gamma = 0$$

If we now take the generator quadric form, & write the condition that the four should lie on the surface, we get the equation of the surface

$$\alpha\gamma = m.\beta\delta.$$

Modifying the constants

$$\alpha\gamma - \beta\delta = 0$$

∴ we have got two systems of generators

$$\left.\begin{array}{l} \alpha - \lambda\beta = 0 \\ \lambda\gamma - \delta = 0 \end{array}\right\} \text{ Ones system}$$

& $$\left.\begin{array}{l} \alpha - \mu\delta = 0 \\ \mu\gamma - \beta = 0 \end{array}\right\} \text{ Second System}$$

To prove that two generators of opposite system intersect:—

From I

$$\frac{\alpha}{\lambda} = \beta \qquad \& \ \gamma = \frac{\delta}{\lambda}$$

From II

$$\frac{\alpha}{\mu} = \delta \qquad \gamma = \frac{\beta}{\mu}$$

$$\therefore \quad \frac{\alpha}{\mu\lambda} = \frac{\beta}{\mu} = \frac{\gamma}{1} = \frac{\delta}{\lambda}$$

which is the point of intersection

This is the parametric representation of the surface.

$\&$ λ = Constant

$\&$ μ = Constant

are two systems of generators.

In the case of the hyperbolic paraboloid which is of the form

$$\frac{y^2}{b} - \frac{z^2}{c} = 2x$$

or $$\left(\frac{y}{\sqrt{b}} - \frac{z}{\sqrt{c}}\right)\left(\frac{y}{\sqrt{b}} + \frac{z}{\sqrt{c}}\right) = 2x$$

We see that the equation is in the form

$$\alpha\gamma = \beta$$

The δ plane is at infinity.

The two system of generators in this case are —

$$\left.\begin{aligned} \alpha &= \lambda\beta \\ \lambda\gamma &= 1 \end{aligned}\right\} \quad - \text{—} \quad \text{I system}$$

$$\left.\begin{aligned} \alpha &= \mu \\ \mu\gamma &= \beta \end{aligned}\right\} \quad - \cdot \quad \text{II system.}$$

Lect. on 5th Nov. 1921 5th Nov. 1921.

The theory of generators admits of purely algebraic treatment thus:-

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

i.e. $$\left(\frac{x}{a} - \frac{z}{c}\right)\left(\frac{x}{a} + \frac{z}{c}\right) = \left(1 - \frac{y}{b}\right)\left(1 + \frac{y}{b}\right).$$

$$\therefore \left.\begin{aligned} \frac{x}{a} - \frac{z}{c} &= \lambda\left(1 - \frac{y}{b}\right) \\ \lambda\left(\frac{x}{a} + \frac{z}{c}\right) &= \left(1 + \frac{y}{b}\right) \end{aligned}\right\} \quad \ldots \quad \mathrm{I}$$

$$\left.\begin{aligned} \frac{x}{a} - \frac{z}{c} &= \mu\left(1 + \frac{y}{b}\right) \\ \mu\left(\frac{x}{a} + \frac{z}{c}\right) &= \left(1 - \frac{y}{b}\right) \end{aligned}\right\} \quad \ldots \quad \mathrm{II}$$

Thus I & II give two systems of generators of the hyperboloid.

Solving we get

$$\frac{\frac{x}{a} - \frac{z}{c}}{\lambda\mu} = \frac{1 - \frac{y}{b}}{\mu} = \frac{\frac{x}{a} + \frac{z}{c}}{1} = \frac{1 + \frac{y}{b}}{\lambda}$$

$\frac{1}{2}$ each $$= \frac{\frac{x}{a}}{1 + \lambda\mu} = \frac{\frac{y}{b}}{\lambda - \mu} = \frac{\frac{z}{c}}{1 - \lambda\mu} = \frac{-1}{\lambda + \mu}$$

$$\therefore \quad x = a\,\frac{1+\lambda\mu}{\lambda+\mu}$$

$$y = b\,\frac{(\lambda-\mu)}{\lambda+\mu}$$

$$\& \quad z = c\,\frac{1-\lambda\mu}{\lambda+\mu}.$$

This is the parametric representation of the surface. Of course

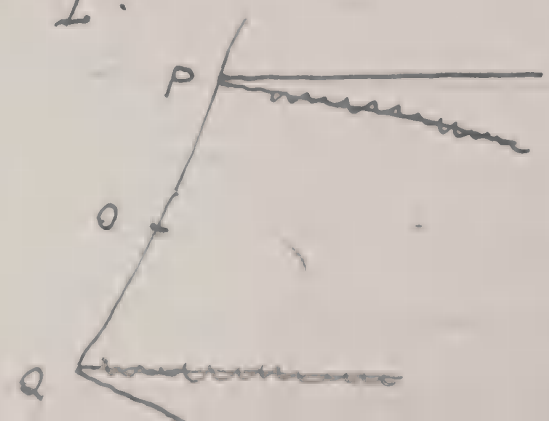
$$\left.\begin{aligned} \lambda &= \text{constant} \\ \& \ \mu &= \text{constant} \end{aligned}\right\} \text{give two sets of}$$

generators.

When $\lambda = -\mu$, we get two generators, belonging to two systems each to each, which are parallel to one another. Because the point of intersection goes to infinity.

We give below various methods of attacking the problems on generators

I.



Through P & Q ~~any~~ two points on the surface

Draw two pairs of parallel generators. Obviously two belong to one system, & the other two respectively parallel through the points belong to another system.

For the sake of clearness we have indicated in figure the generators of the two ~~same~~ systems differently.

Join PQ, & through O the mid point of PQ take axes parallel to the generators.

We have now oblique axes through O the centre of the surface.

The equation of the surface now reduces to the form

$$z^2 + 2hxy = c^2$$

I

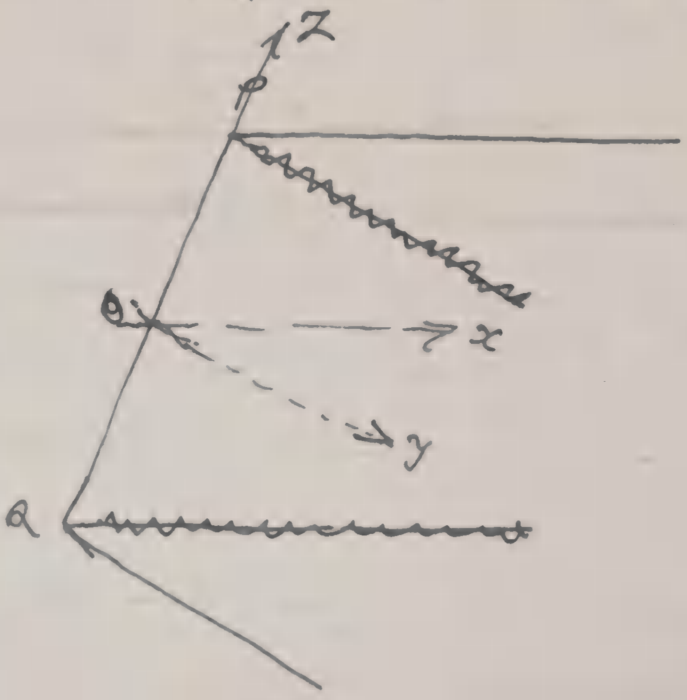
Through P any point on the surface take two generators one of each system.

Join PO, where O is the centre of the surface, & Produce it to meet it again in Q

Then OP = OQ - -.

Now, the plane through P, passing through the generators is a tangent plane at P

Similarly the tangent plane at Q cuts the surface in two generators which are respectively parallel to the first pair.



Taking now the axis of z along PQ and, lines through O, parallel to these generators as axes of x & y, the equation of the surface reduces to the form

$$z^2 + 2hxy = c^2$$

which can be written in the form

$$2hxy = (c-z)(c+z)$$

$\therefore$ The two systems of generators are given by

$$\left.\begin{aligned} x &= \lambda(c-z) \\ 2h\lambda y &= c+z) \end{aligned}\right\} \quad \mathrm{I}$$

$$\left.\begin{aligned} x &= \mu(c+z) \\ 2h\mu y &= (c-z) \end{aligned}\right\} \quad \underline{\mathrm{II}}.$$

This system of Axes can be applied in the following example:—

· Show that if two generators of opposite system be taken, & points P, Q on them are taken such that the generators through them are parallel, then PQ.

touches a hyperbola of which the two first generators are asymptotes.

Show also that this hyperbola is the conjugate one to the parallel central section of the hyperboloid.

// If a, b, c d be four generators of one system, & p a generator of the second system, which cuts these four, we have then got four planes viz (p, a) (p, b) (p, c) (p, d).

Now any other generator of the opposite system, say q were to intersect, like p, a, b, c, d, it will cut these in points where it passes across the four planes stated above.

∴ The anharmonic ratio of the four points is constant.

Stated in another way we have:—

If we take two fixed generators of one system.

Then the generators of the other system cut these two in points which have projective property.*

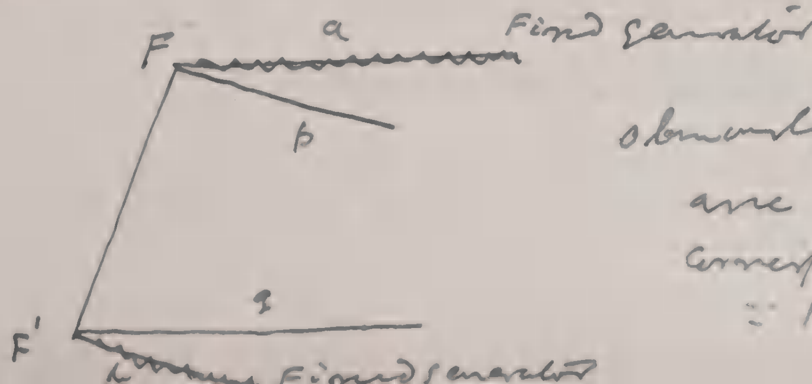
[* Cf:- the following propositions in Askwith's Pure Geometry:—

1. Two ranges $ABCDE$ - - $A'B'C'D'$- — are said to be homographic, when a cross ratio of any four points of the one is equal to the corresponding cross ratio of the four corresponding points of the other.

And further

2. Two homographic pencils are mutually projective].

If F & F' are points the two generators, corresponding to ∞ on the other. $FP \cdot F'P' =$ constant.

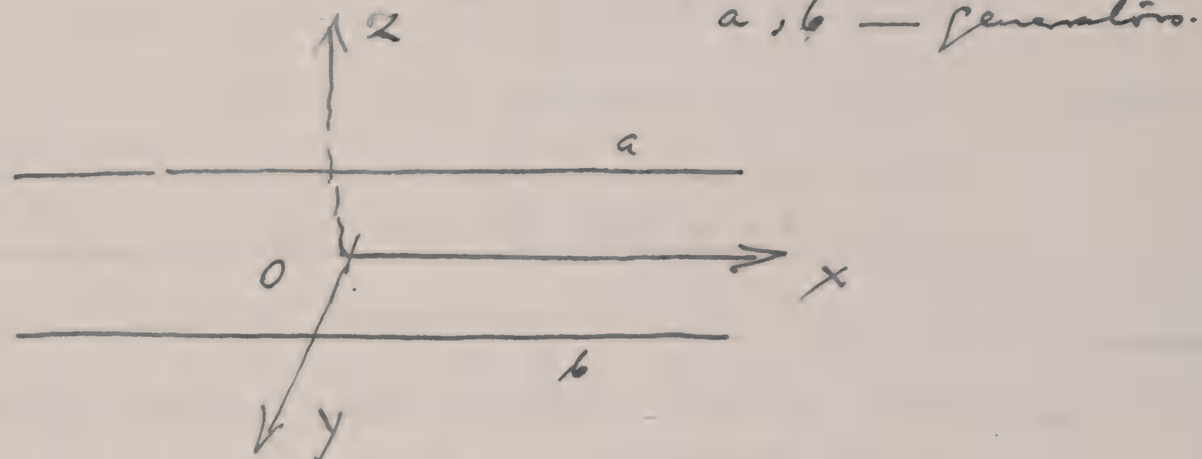


obviously F & F' are the points corresponding to ∞ $\therefore p \parallel b$ & $q \parallel a$.

II.

Choose two parallel generators, one of each system of course.

Let the plane determined by these be taken as the xy plane



Let $2p$ = shortest distance (a, b).

If we take the equation to the surface in the form

$$ax^2 + y^2 + cz^2 + 2hxy + 2gzx + 2fyz = C$$

This must reduce to $y^2 = p^2$ when $z = 0$

$$\therefore a = h = 0 \quad \& \quad C = p^2$$

$\therefore$ The equation becomes

$$y^2 + cz^2 + 2fyz + 2gzx = p^2$$

i.e. $z(2gx + 2fy + cz) = (p-y)(p+y)$.

And the two systems of generators can be taken as

$$\left.\begin{aligned} z &= \lambda(p-y) \\ \lambda(2gx+2fy+cz) &= p+y \end{aligned}\right\} \quad \mathrm{I}$$

$$\left.\begin{aligned} z &= \mu(p+y) \\ \mu(2gx+2fy+cz) &= p-y \end{aligned}\right\} \quad \mathrm{II}.$$

Example:— Show that, along a generator, the normal to the surface turns round, as we pass along the generator in such a way that the positions at which they are at rt. angles form an involution with a constant ~~in~~ negative of involution.

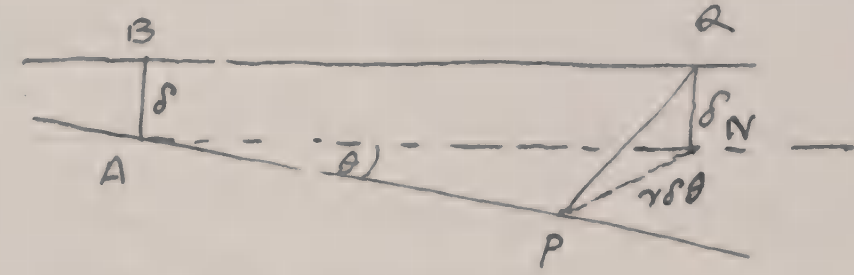
Determine the co-ordinates of the centre of the involution, & interprete the value of the constant in terms of the constants of the hyperboloid.

Lect. on 8th Nov. 1921 8th Nov. 1921

It is easy to see that at the tangent plane at the point, which is the centre of involution, passes through the generator which contains the range in involution.

The locus of the centres of involution is called the Line of Striction.

"Line of Striction on any ruled surface" — in which no two consecutive generators intersect.



Let AP be a generator & BQ the consecutive one.

It is supposed that $Lt. \frac{\delta}{\theta}$ is not zero but is finite limit say ϖ.

Consider the tangent plane at P. It must contain generator AP. And to fix it consider the other tangent line $\perp$ AP, &

which must cut BQ in say Q.

Then if $AP = r$; $PN = r\delta\theta$; $QN = \delta$

The normal to the surface perpendicular to the generator PQ, must make with the ultimate direction of AB an angle

$$\tan^{-1}\left[\text{Lt.}\ \frac{\delta}{r\delta\theta}\right]$$

$$\text{i.e.}\ \tan^{-1}\frac{\varpi}{r}.$$

At the point A, normal $\perp$ AB

At infinity normal is $\parallel$ AB

In order that the normals should be at rt angles, we must have the product of their slopes $= -1$

$$\text{i.e.}\quad \frac{\varpi}{r}\cdot\frac{\varpi}{r'} = -1$$

$$\therefore\ rr' = -\varpi^2.$$

which shows that the points are in involution, & A is the centre of involution. Moreover, the constant of involution is $-$ve.

If we ~~find to~~ want to find the line of Striction, we proceed thus:—

In the general case

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

There are two branches of the line-

We have seen that the two sets of generators are given by

$$\frac{x - a\cos\alpha}{a\sin\alpha} = \frac{y - b\sin\alpha}{-b\cos\alpha} = \frac{z}{\pm c}.$$

Any point on a generator of the 1st system, (when +ve sign is taken) is given as

$$\left\{ a\left(\cos\alpha + \sin\alpha\frac{z}{c}\right), b\left(\sin\alpha - \cos\alpha\frac{z}{c}\right), z \right\}.$$

And the direction cosines of the normal at this point are proportional to

$$\frac{1}{a}\left(\cos\alpha + \sin\alpha\frac{z}{c}\right), \frac{1}{b}\left(\sin\alpha - \cos\alpha\frac{z}{c}\right), -\frac{z}{c^2}.$$

Thus if z_1, z_2 are the two points normals at which are $\perp$ each other we get

$$\frac{1}{a^2}\left(\cos\alpha + \sin\alpha\frac{z_1}{c}\right)\left(\cos\alpha + \sin\alpha\frac{z_2}{c}\right)$$

$$+ \frac{1}{b^2}\left(\sin\alpha - \cos\alpha\frac{z_1}{c}\right)\left(\sin\alpha - \cos\alpha\frac{z_2}{c}\right) + \frac{z_1 z_2}{c^4} = 0 \quad \text{I}$$

Now this can be written in the form

$$L z_1 z_2 + M(z_1 + z_2) + N = 0$$

or $$\left(z_1 + \frac{M}{L}\right)\left(z_2 + \frac{M}{L}\right) = \frac{M^2 - NL}{L^2}$$

Thus we find the centre of the involution is at the point

$$z = -\frac{M}{L}.$$

To simplify we see that I can be written in the form

$$z_1 z_2 \left[\frac{1}{c^4} + \frac{1}{c^2}\left(\frac{\sin^2\alpha}{a^2} + \frac{\cos^2\alpha}{b^2} \right) \right]$$

$$+ (z_1 + z_2)\left[\frac{\sin\alpha\cos\alpha}{c}\left(\frac{1}{a^2} - \frac{1}{b^2} \right) \right] + \frac{\cos^2\alpha}{a^2} + \frac{\sin^2\alpha}{b^2}$$

$$\therefore -\frac{M}{L} = \frac{\left(\frac{1}{b^2} - \frac{1}{a^2} \right) \frac{\sin\alpha\cos\alpha}{c}}{\frac{1}{c^4} + \frac{1}{c^2}\left(\frac{\sin^2\alpha}{a^2} + \frac{\cos^2\alpha}{b^2} \right)}$$

$$= \frac{c\sin\alpha\cos\alpha\left(\frac{1}{b^2} - \frac{1}{a^2} \right)}{\frac{1}{c^2} + \frac{\sin^2\alpha}{a^2} + \frac{\cos^2\alpha}{b^2}}$$

$$= \frac{c\sin\alpha\cos\alpha\left(\frac{1}{b^2} - \frac{1}{a^2} \right)}{\sin^2\alpha\left(\frac{1}{a^2} + \frac{1}{c^2} \right) + \cos^2\alpha\left(\frac{1}{b^2} + \frac{1}{c^2} \right)}$$

$$= \frac{c\sin\alpha\cos\alpha\,(B-A)}{A\sin^2\alpha + B\cos^2\alpha}$$

where $A = \frac{1}{a^2} + \frac{1}{c^2}$

& $B = \frac{1}{b^2} + \frac{1}{c^2}$

Of course the x & y coordinates are obtained

from $x = a(\cos\alpha + \sin\alpha \frac{z}{c})$

& $y = b(\sin\alpha - \cos\alpha\, z/c)$.

The result is a curve of which the coordinates are given in a parameter α

$$\left.\begin{aligned} x &= f_1(\alpha) \\ y &= f_2(\alpha) \\ z &= f_3(\alpha) \end{aligned}\right\}$$

Putting $t = \tan\frac{\alpha}{2}$, we easily see that the branch is of the fourth degree.

The other branch corresponding to the second set of generators is given by changing c into $-c$

And hence it is simply the reflection of the first branch in the principal.

Examples for homework:—

(1) If on any two fixed lines sets of points are taken which are mutually projective, then the line joining corresponding points generates a hyperboloid.

(2) If three generators all parallel to the same plane be taken, show that the family of lines which intersects these three generate a hyperbolic paraboloid.

If a line meets three generators of the other system in P, Q, R

$$PQ/PR = \text{Constant}.$$

(3) To an eye situated outside a hyperboloid of one sheet, every generating line will appear to lie on another.

If it is placed on the point (fgh) Prove that the points on the surface the generating lines through which appear to be perpendicular will be in the plane

$$(a+b+c)(afx + bgy + chz - 1) = 2(a^2fx + b^2gy + c^2hz).$$

Lect on 10th nov. 1921 10th nov. 1921

"Confocal Conicoids"

1°. We start with the analytical definition of a system of confocals viz.

$$\Sigma \frac{x^2}{a^2+\lambda} = 1$$

2° Through any point there pass three confocals. For if x, y, z be the point

$$\Pi (a^2+\lambda) - \Sigma x^2 (b^2+\lambda)(c^2+\lambda) = 0$$

$$\equiv (\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3) = 0$$

this equation gives three values of λ.

Putting $\lambda = -a^2, -b^2, -c^2$

the co-ordinates of the points are obtained, as

$$x^2 = \frac{(a^2+\lambda_1)(a^2+\lambda_2)(a^2+\lambda_3)}{(a^2-b^2)(a^2-c^2)}$$

$$y^2 = \frac{(b^2+\lambda_1)(b^2+\lambda_2)(b^2+\lambda_3)}{(b^2-c^2)(b^2-a^2)}$$

$$z^2 = \frac{(c^2+\lambda_1)(c^2+\lambda_2)(c^2+\lambda_3)}{(c^2-a^2)(c^2-b^2)}.$$

Also from the original, from the sum of the roots we get

$$x^2+y^2+z^2 = (a^2+b^2+c^2) + (\lambda_1+\lambda_2+\lambda_3).$$

Thus the co-ordinates are expressed in terms of the semi diameters of the confocals.

3°. At P let the two normals to the two confocals with the first, be drawn. Then they must lie in a tangent plane at P to the original conicoid. For, the three conicoids which are confocal with each other cut orthogonally; & hence the normals are mutually perpendicular at their common point of intersection.

If $P(x, y, z)$ be the point & a_2, a_3 the semi diameters of the other two conicoids; the direction cosines of normals are

$$\frac{p_2 x}{a_2^2}, \quad \frac{p_2 y}{b_2^2}, \quad \frac{p_2 z}{c_2^2}$$

and

$$\frac{p_3 x}{a_3^2}, \quad \frac{p_3 y}{b_3^2}, \quad \frac{p_3 z}{c_3^2}.$$

Now these two directions are mutually perpendicular.

$$\therefore \Sigma \frac{x^2}{a_2^2 a_3^2} = 0$$

But this ~~[illegible]~~ leads to the condition of conjugacy of the two lines. For, —

If the lines are to be conjugate the necessary and sufficient condition is in the present case

$$\Sigma \frac{x^2}{a_1^2 \cdot a_3^2} = 0$$

This is immediately obtained by the fact

$$\Sigma \frac{x^2}{a_1^2 a_2^2} = 0 \quad \& \quad \Sigma \frac{x^2}{a_1^2 a_3^2} = 0$$

Subtracting we get

$$\Sigma \frac{x^2 (a_3^2 - a_2^2)}{a_1^2 a_2^2 a_3^2} = 0$$

i.e. $\quad \Sigma \frac{x^2}{a_1^2 a_2^2 a_3^2} = 0$

since $a_3^2 - a_2^2 = b_3^2 - b_2^2 = c_3^2 - c_2^2$.

Thus the normals to the two confocals at the common point of intersection with the first, are conjugate, at right angles to each other, and lie in a tangent plane to the 1st at that point.

It follows, therefore, that the central parallel section has its diameters, which are parallel to these two normals, both conjugate & at right angles.

They are, therefore its Principal Axes

4° To find the lengths of these Axes:-

We know if r is the length & l, m, n the direction cosines.

$$\frac{1}{r^2} = \frac{l^2}{a_1^2} + \frac{m^2}{b_1^2} + \frac{n^2}{c_1^2}$$

∴ The diameter parallel to the normal to the a_2-surface is

$$\frac{1}{r^2} = \frac{p_2^2 x^2}{a_1^2 a_2^4} + \frac{p_2^2 y^2}{b_1^2 b_2^4} + \frac{p_2^2 z^2}{c_1^2 c_2^4}.$$

but $$\frac{1}{p_2^2} = \Sigma \frac{x^2}{a_2^4}$$

$$\& \Sigma \frac{x^2}{a_1^2 a_2^2} = 0$$

$$\therefore \frac{1}{p_2^2} = \Sigma \frac{x^2 (a_1^2 - a_2^2)}{a_1^2 a_2^4}$$

$$\therefore \frac{1}{a_1^2 - a_2^2} = \Sigma \frac{p_2^2 x^2}{a_1^2 a_2^4}$$

which shows that

$$r^2 = a_1^2 - a_2^2$$

Similly the other semi-axis $= a_1^2 - a_3^2$.
This is a solution in another form of the problem of finding the lengths of a section of a Conicoid.

5° ~~As~~ a corollary to the above ~~the~~ result, we deduce that

$$p_1^2 = \frac{a_1^2 b_1^2 c_1^2}{(a_1^2-a_2^2)(a_1^2-a_3^2)}$$

$$p_2^2 = \frac{a_2^2 b_2^2 c_2^2}{(a_2^2-a_1^2)(a_2^2-a_3^2)}$$

$$p_3^2 = \frac{a_3^2 b_3^2 c_3^2}{(a_3^2-a_1^2)(a_3^2-a_2^2)}.$$

Corollary II :- Let us suppose that the conicoid a_1 is cut by the conicoid a_2. Then for every point on the curve of intersection a_1, a_2 are the same though of course a_3 is different.

$$\therefore\ p_1^2(a_1^2-a_3^2) = \frac{a_1^2 b_1^2 c_1^2}{a_1^2-a_2^2}$$

$$= \text{Constant}.$$

but $a_1^2-a_3^2$ = the sq. on the semi-diameter parallel to ~~the~~ a_3-normal i.e. parallel to the ~~tl~~ tangent to the curve of intersection of $(a_1 \,\&\, a_2)$.

The result is often stated in the form p^2D^2 = Constant, D being the diameter parallel to the tangent plane & p the perpendicular

[illegible]

on the tangent plane.

$\therefore$ 25th Dec. 1922

In general suppose we have to find the condition that the lines

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \qquad (i)$$

$$\& \quad \frac{x-\alpha'}{l'} = \frac{y-\beta'}{m'} = \frac{z-\gamma'}{n'} \qquad (ii)$$

should be conjugate with respect to the conicoid

$$a^{-2}x^2 + b^{-2}y^2 + c^{-2}z^2 = 1$$

The polar plane of any point $(\alpha + lr, \beta + mr, \gamma + nr)$ on (i) with respect to the conicoid $\Sigma x^2/a^2 = 1$ is,

$$\Sigma \frac{x(\alpha + lr)}{a^2} = 1$$

i.e. $$\Sigma \frac{x\alpha}{a^2} - 1 + r\Sigma \frac{lx}{a^2} = 0$$

& this evidently passes for all values of r through the line

$$\left.\begin{aligned} \Sigma \frac{x\alpha}{a^2} &= 1 \\ \Sigma \frac{lx}{a^2} &= 0 \end{aligned}\right\} (iii)$$

Hence this must be identical with (ii)

For this first $(\alpha' \beta', \gamma')$ must satisfy the two planes in (iii), & the $(l' m' n')$ direction is $\perp$ to normals to both

$\therefore$ We must have
$$\left\{\begin{aligned} &\Sigma \frac{\alpha\alpha'}{a^2} = 1 \\ &\Sigma \frac{l\alpha'}{a^2} = 0 \\ &\Sigma \frac{l'\alpha}{a^2} = 0 \quad \& \quad \Sigma \frac{ll'}{a^2} = 0 \end{aligned}\right.$$

4 Conditions.

All these conditions are satisfied in the present case considered.

Lecture. 12th Nov. 1921 — 13th nov. 1921.

6° Any plane touches a confocal of the system, & the normal at the point of contact is the locus of poles of the plane with respect to all the confocals.

Proof:—

If $lx+my+nz-p=0$ were to touch a member of

$$\Sigma \frac{x^2}{a^2+\lambda} = 1$$

Then $\Sigma (a^2+\lambda) l^2 - p^2 = 0$

$\therefore$ The λ is given by

$$\lambda = p^2 - \Sigma a^2 l^2 = \lambda_1 \text{ say.}$$

Calling this particular confocal touched by the plane by 'λ_1-confocal'

The pole of this plane with respect to any conic is

$$(x\,y\,z) = \left\{ \frac{(a^2+\lambda)l}{p}, \frac{(b^2+\lambda)m}{p}, \frac{(c^2+\lambda)n}{p} \right\}$$

and this becomes the point of contact if $\lambda = \lambda_1$

Now $x = \dfrac{(a^2+\lambda)l}{p} = \dfrac{(a^2+\lambda_1)l}{p} - \dfrac{(\lambda_1-\lambda)l}{p}$

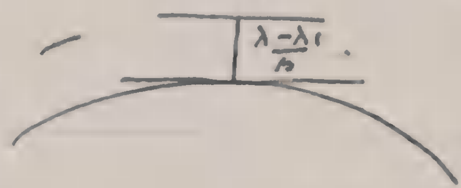
$y = \dfrac{(b^2+\lambda)m}{p} = \dfrac{(b^2+\lambda_1)m}{p} - \dfrac{(\lambda_1-\lambda)m}{p}$ &

i.e.

Showing that as λ varies, the point (x, y, z) runs along a st. line passing through the first point (of contact), and being at rt. angles to the plane.

$\therefore$ The normal at the point is the locus. Moreover:

$\frac{\lambda - \lambda_1}{p}$ is the distance from a pole to the point of contact.



7°. The Axes of the enveloping cone drawn from a point to one of the confocals are the normals to the three confocals that pass through the point.

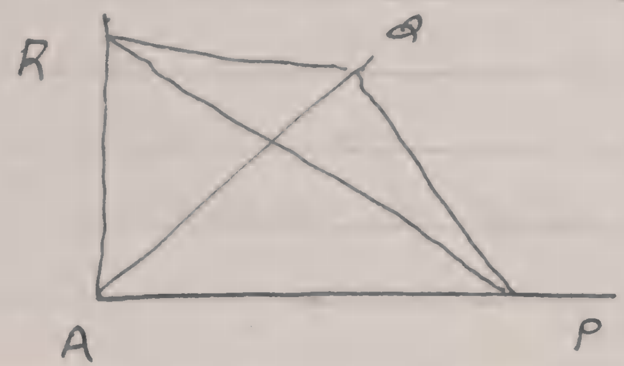
And the equation of the cone referred to these axes is

$$\frac{x^2}{\lambda_1} + \frac{y^2}{\lambda_2} + \frac{z^2}{\lambda_3} = 0$$

where $\lambda_1 = a^2 - a_1^2$; $\lambda_2 = a^2 - a_2^2$; $\lambda_3 = a^2 - a_3^2$.

where a is the semi axes of the one which is touched by the cone; & $a_1\, a_2\, a_3$ being those of the three confocals through the point.

Proof:- Let A be the vertex of the cone; and let the three normals to the three confocals cut the plane of contact of the cone in P, Q, R.



We know from the preceding proposition that the lengths are

$$AP = \lambda_1 / p_1$$
$$AQ = \lambda_2 / p_2$$
$$AR = \lambda_3 / p_3$$

[Because, the line AP is the locus of the poles of of the plane AQR . . &c]

P is the pole of of plane AQR with respect to the special confocal touched by the cone.

Similar Q & R.

It follows that PQR must be a self conjugate triangle with regard to the

the conic in which the plane of contact intersects the confocal & the enveloping cone.

$\therefore$ AP, AQ, AR are the conjugate lines of the cone.

Moreover, they are mutually perpendicular

$\therefore$ They are the Principal axes of the cone.

The equation of the cone referred to them must be of the form

$$Ax^2 + By^2 + Cz^2 = 0.$$

The line from A to the common centre of all the confocals is conjugate to the conic of contact

Again the coordinates of the centre of the conicoid, referred to these axes are (p_1, p_2, p_3)

$\therefore$ The equation to the polar plane is

$$Axp_1 + Byp_2 + Czp_3 = 0.$$

But this must be parallel to the plane PQR, which is, as referred to the new axes

$$\Sigma \frac{x}{\lambda/p_1} = 1$$

The parallelism of these two planes gives

$$A/\lambda_1^{-1} = B/\lambda_2^{-1} = C/\lambda_3^{-1}$$

∴ The equation to the cone becomes

$$x^2/\lambda_1 + y^2/\lambda_2 + z^2/\lambda_3 = 0$$

or, in other words

$$\Sigma\, x^2/(a^2-a_1^2) = 0$$

If we vary a we see that the different enveloping cones from the point (a_1, a_2, a_3) are confocal cones because we have merely to add a constant σ to a^2.

8° Any St. line touches two of the Confocals: and the two corresponding tangent planes are at right angles to each other.

Proof:– Take any point of the line, & let its elliptic co-ordinates be $(a_1\, a_2\, a_3)$. And let (l, m, n) be the direction cosines of the line as referred to the the three normals at the point.

P. T. O.

Then if the line touches the 'a'- confocal, it must lie on the cone

$$\frac{x^2}{a^2-a_1^2} + \frac{y^2}{a^2-a_2^2} + \frac{z^2}{a^2-a_3^2} = 0$$

∴ We must have

$$\frac{l^2}{a^2-a_1^2} + \frac{m^2}{a^2-a_2^2} + \frac{n^2}{a^2-a_3^2} = 0$$

Considered as a quadratic in a^2, we get two values, & thus there are two confocals touched by a line.

// To prove the latter part, let us simplify its geometry by taking the origin at one of the points of contact.

Suppose that the line touches 'a_1' – confocal, & a_2, a_3 are the coordinates of the point of contact.

Now we know, that the system of enveloping cones to different confocals from the vertex A have the same axes — viz. the normals at the pt. A to 'a_1-a_2-a_3' confocals. And the given line lies in one of the principal planes

If we take any confocal & draw tangent planes to it passing through AB, these two tangent planes, (since they touch the enveloping cone for that confocal) must be equally inclined in the opposite senses to the plane AYZ.

These two tangent planes will coincide ~~in two cases~~

~~(i)~~ When the line AB is itself a tangent line to the confocal

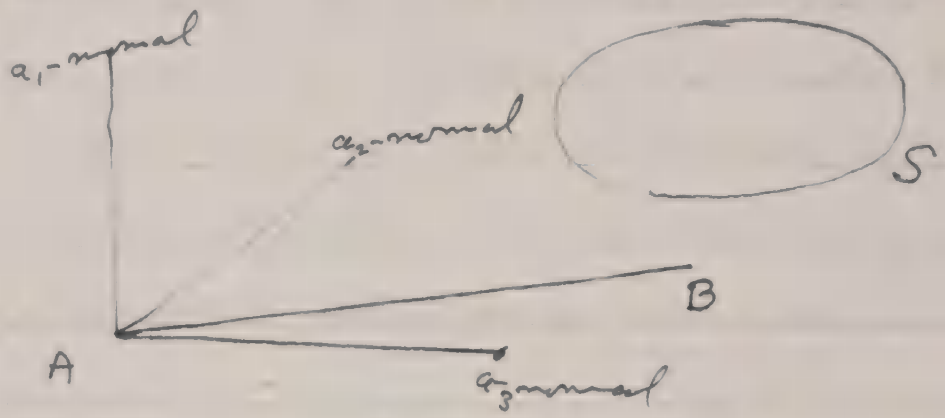
which happens ~~often~~ in two cases:-

(i) one case of coincidence being obtained when the planes coincide with pl. AYZ.

(ii) the other being obtained when they coincide with the other plane bisecting the angle between the pairs of tangent planes.

∴ The other confocal is touched by the plane ⊥ that touching the 1st.

Note :— 25th Dec. 1922.



The line AB touches a_1-confocal

∴ It lies in the plane of $(\overline{A - a_2 - a_3})$.

Now, suppose, AB is to touch S-confocal

Now, ordinarily, there are two planes through AB which touch S, & they will coincide when AB itself touches S.

∴ The condition of AB touching S is the coincidence of two tangent planes through AB.

The two tangent-planes ordinarily touch the enveloping cone of S, vertex A, & since AB lies in ~~the~~ a principal plane, the tangent planes are equally inclined to it, ... etc

∴ The conclusion is Any st. line touches two of the confocals, & the corresponding tangent planes are either (i) Coincident

or (ii) Perpendicular.

9°. If a point be taken on an ellipsoid; the locus of the corresponding points on the confocals is the curve of intersection of two hyperboloids of the system; which pass through that point.

<u>Definition</u>; — Corresponding points.

If $(a\,b\,c)$ $(a'\,b'\,c')$ are two confocals and points (x, y, z) $(x'\,y'\,z')$ are taken on them such that

$$\frac{x}{a} = \frac{x'}{a'}\,;\quad \frac{y}{b} = \frac{y'}{b'}\,;\quad \frac{z}{c} = \frac{z'}{c'}\,.$$

The points are said to correspond.

Proof:— The co-ordinates of a point can be expressed in ~~the form~~:- -the following manner:—

Suppose (a, b, c) is the fixed ellipsoid. And $(a'\,b'\,c')$ is a varying one of the system and let two others of the system cut $(a'\,b'\,c')$ in the point whose elliptic co-ordinates are $(a', a_2\; a_3)$.

Then by what we have seen in 2°;

$$x'^2 = \frac{a'^2 a_2^2 \cdot a_3^2}{(a^2-b^2)(a^2-c^2)} \quad \&c.$$

$$\therefore \frac{x'^2}{a'^2} = \frac{a_2^2 \cdot a_3^2}{(a^2-b^2)(a^2-c^2)} = \text{constant}$$

which shows that the locus of $(x' y' z')$ which is now a now corresponding point in all its positions ($\because x'/a'$ ~~&c~~ are constant) is the curve of intersection of 'a_2, a_3'—confocals.

But of the three confocals through the point, if as we have supposed 'a'' confocal is the ellipsoid, the other two viz (a_2, a_3) must be hyperboloids of one sheet & two sheets.

$\therefore$ The curve of intersection of two hyperboloids of the system intersects the confocal ellipsoids in corresponding points.

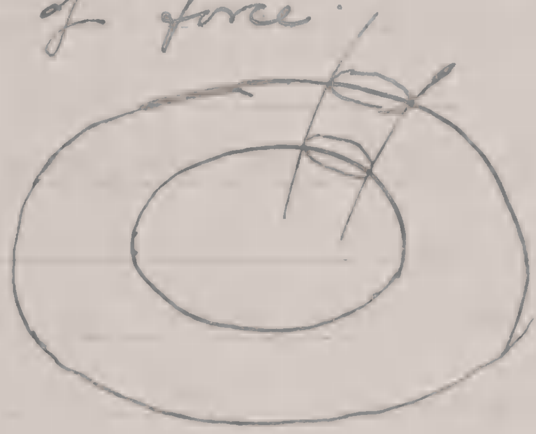
Further, confocals cut at right angles

$\therefore$ These curves of intersection also cut every confocal orthogonally.

This property is important in Physical Analysis :-

We know that equi-potential Surfaces are cut orthogonally by the lines of force.

Thus the confocal ellipsoids are equipotential surfaces, when the curves of intersection in question are lines of force.



10°. Ivory's Theorem :— If (P, P') & (Q, Q') are two pairs of corresponding points on two confocals then

$$PQ' = P'Q.$$

The proof is easy.

Now we come to the most important Theorem about Con-focals.

Confocal Conicoids are all inscribed in the same developable surface. — A surface which has the focal conic as the nodal line.

The tangential equation to a system of confocals

$$\Sigma \frac{x^2}{a^2+\lambda} = 1$$

is $\Sigma (a^2+\lambda) l^2 - p^2 = 0$

where $lx + my + nz - p = 0$ is the tangent plane.

And the equation can be written in the form

$$\Sigma a^2 l^2 - p^2 = \lambda (l^2 + m^2 + n^2).$$

∴ It follows that the planes defined by

$$\left.\begin{array}{ll} \Sigma a^2 l^2 - p^2 = 0 & \ldots (i) \\ \& \quad \Sigma l^2 = 0 & \ldots (ii) \end{array}\right\}$$

touch every confocal of the system.

But two equations of the type define a 'developable' Surface, with one

infinity of tangent planes. (Note:— This is usually written as ∞' tangent planes).

Moreover (i) means that the plane touches a definite conicoid of of the system.

and (ii) means that the plane touches the imaginary circle at infinity.

This Developable, therefore, has all the con-focals inscribed in it.

"Examples for homework":-

1. Show that if we take a point as in the theory of tangent cones, the cone of the normals which can be drawn from that point to all the confocals is

$$\Sigma \frac{p_1(\lambda_2 - \lambda_3)}{x} = 0$$

2. If through a fixed line planes be drawn to touch confocals, show that the corresponding normals lie on a hyperbolic paraboloid.

3. A ray of light is reflected continuously at the surface of one of the confocals. Show that the various st. lines of its path touch two fixed confocals.

Lect. on 15th Nov. 1921 15th Nov. 1921.

We have seen that the tangent planes to the system of confocals satisfy

$$\Sigma a^2 l^2 - p^2 = 0$$

$$\& \quad \Sigma l^2 = 0$$

These are the condition that the plane touches a special confocal & the circle at infinity.

$\therefore$ The plane has only one degree of freedom.

It therefore generates a surface & touches it all along a st. line. Such a surface is a developable surface.

$\therefore$ All the confocal conicoids are contained in the same developable surface.

The three focal conics, two real & one imaginary, are touched by the planes, as also the ⊙le at infinity.

Proof:— $\Sigma a^2 l^2 - p^2 = 0$
$\& \ \Sigma l^2 = 0$

$$\therefore (a^2-c^2) l^2 + (b^2-c^2) m^2 - p^2 = 0$$

and this is the tangential equation of that

focal conics.

Take any one of the focal conics & consider a tangent line to it, & consider the circle at infinity.

Through the tangent line to the focal conic, it is possible to draw two planes which touch the the ⊙le at infinity:

∴ Through this point there will pass two distinct generators of this developable surface. And this is true for every point as we pass along the focal conics.

Such a line on a surface is called a nodal line.

In other words two different sheets of the developable surface intersect one another along the focal conic.

The same is true for the other two focal conics.

Lect on: 17th Nov. 1921 18th Nov. 1921.

Bifocal chords.

Let us suppose that of the system of confocals $\Sigma \frac{x^2}{a^2+\lambda} = 1$ a fixed one is taken say

$$\frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} + \frac{z^2}{c_1^2} = 1$$

Take any point P on it defined by the values a_2, a_3 the ~~normals~~ semi major axes of the other two confocals passing through it.

And let a chord PQ be drawn of the confocal.

Let the direction cosines of PQ referred to the normals to the three confocals at the point be (l, m, n); — l being that with the normal to the original 'a_1' – confocal.

Then the equation

$$\frac{l^2}{a_1^2-a^2} + \frac{m^2}{a_2^2-a^2} + \frac{n^2}{a_3^2-a^2} = 0$$

is, as we have so often noted, a quadratic in a^2 giving the two confocals which are touched by the line l, m, n i.e. the chord PQ.

We suppose that, for different positions of P, PQ is drawn in such a way that it touches the same two confocals.

Let α_1^2, α_2^2 be the values of a^2 as given by the quadratic.

Then we may easily write

$$\Sigma\, l^2(a^2-a_2^2)(a^2-a_3^2) \equiv (a^2-\alpha_1^2)(a^2-\alpha_2^2) = 0$$

Remembering that the coefficients of the highest term on the left hand side is $(l^2+m^2+n^2)$ i.e. 1.

Putting $a^2 = a_1^2$ in the identity we get

$$l^2 = \frac{(a_1^2-\alpha_1^2)(a_1^2-\alpha_2^2)}{(a_1^2-a_2^2)(a_1^2-a_3^2)}.$$

Now if p_1 be the perpendicular from centre on the tangent plane at P to the 'α_1'-confocal,

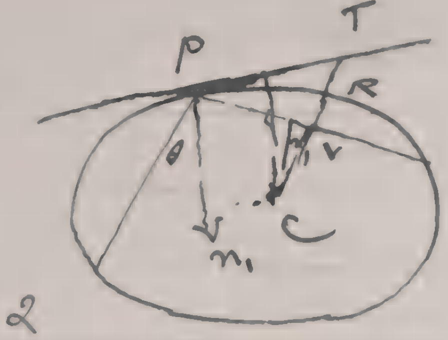
$$p_1^2 = \frac{a_1^2 b_1^2 c_1^2}{(a_1^2-a_2^2)(a_1^2-a_3^2)}.$$

[Since $a_1^2-a_2^2$, $a_1^2-a_3^2$ are the squares of the semiaxes of the central section parallel to the tangent plane].

$$\therefore \frac{p_1^2}{l^2} = \frac{a_1^2 b_1^2 c_1^2}{(a_1^2-\alpha_1^2)(a_1^2-\alpha_2^2)}.$$

If then $\alpha_1\, \alpha_2$ are constant for any chord PQ

$\dfrac{p_1}{l}$ is also constant.



[$Pn_1 \rightarrow$ normal to a_1-confocal.

$\cos\theta = l$].

$\therefore$ CT $(= p_1/l)$ is constant.

i.e. the length of chord PQ which is equal to $2\,CR^2/CT$ $(=2CV)$.

<u>varies</u> as the square on parallel semi-diameter.

One important case occurs, when for the two fixed confocals α_1, α_2 which are touched by the chord PQ in all its positions, we choose the two real focal conics — the focal ellipse & the focal hyperbola. In this case PQ will pass through their edges.

In this case

$$\alpha_1^2 = a_1^2 - c_1^2$$

$$\alpha_2^2 = a_1^2 - b_1^2$$

In this case

$$\therefore\ CT^2 = \frac{p_1^2}{l^2} = \frac{a_1^2 \cdot b_1^2 c_1^2}{\{a_1^2 - (a_1^2 - c_1^2)\}\{a_1^2 - (a_1^2 - b_1^2)\}} = a_1^2$$

$\therefore CT = a_1$

The length of the chord PQ in this case is equal to $\dfrac{2CR^2}{a_1}$

$$PQ = \frac{2CR^2}{a_1}$$

Such a chord (PQ) is called a byi-focal chord. (It passes through the edges of the two real focal conics).

Confocal cones.

In the case of a family of confocal cones

$$\Sigma \frac{x^2}{a+\lambda} = 0$$

replacing $a^2\ b^2\ c^2$ by $a\ b\ c$ in order that one of the quantities may be -ve, we find that the focal conics, are reduced to a pair of st. lines

$$\frac{x^2}{a-b} = \frac{z^2}{b-c} \qquad \because a > b > c.$$

The focal ellipse gives imaginary lines; the the focal hyperbola reducing in this case to a pair of st. lines.

These are called the focal lines of the system of confocal cones.

The sections at right angles to either

of the focal lines must be such that the point of intersection must be the focus of the section.

Prove that the planes giving the circular sections of the system of reciprocal cones of the confocal system, cut at right angles one or other of the focal lines.

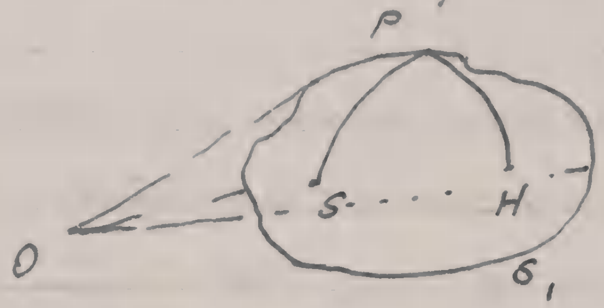
Deduce geometrically that the section of the original cone by the plane is the polar reciprocal of the circle with regard to the point on the focal line as the origin of reciprocation. It follows that the point must be a focus.

"Sphero - Conics".

// The theory of cones may also be studied from their intersection with a concentric unit sphere. Obviously the cone will intersect the sphere in a curve. Such curves are called sphero-conics, — if the cone is a quadric cone (of the second degree). They are generally of the fourth degree. But they can be simplified; as they have an exact counterpart on the other side of the sphere.

The focal lines pierce the sphere through two points S, H which we called

the foci. The is appropriately given as will be obvious from the following consideration:—



If S_1 is the curve of intersection on the sphere, & P a point in which ~~the~~ a generator of the cone cuts it, P is related to S & H in such a way, the ~~great~~ arcs of great circles from P to S & H are together constant.

$$PS + PH = \text{Constant}$$
$$= \cos^{-1} \frac{a+c}{a-c}.$$

[c of course has got to be −ve for the existence of real conic].

Lect on 19th nov. 1921 19th nov. 1921

"Solutions of Examples"

1. A point is taken within an ellipsoid. To show that it is the focus of two real sections:-

Let (α, β, γ) be the point in the ellipsoid $\Sigma \frac{x^2}{a^2} = 0$

and let $\lambda_1\ \lambda_2\ \lambda_3$ be the parameters of the three confocals passing through the point.

Obviously, since the point is within, the ellipsoid $\lambda_1\ \lambda_2\ \lambda_3$ are all −ve quantities, & if in descending order of magnitude represent ellipsoid, hyperboloid of one sheet, & hyperboloid of two sheets respectively.

The enveloping cone from (α, β, γ) to the original ellipsoid is

$$\left(\Sigma \frac{\alpha^2}{a^2} - 1\right)\left(\Sigma \frac{x^2}{a^2} - 1\right) = \left(\Sigma \frac{x\alpha}{a^2} - 1\right)^2$$

Of course the cone must be imaginary as the point is inside.

If referred to the normals to the three confocals at the point, the equation becomes:

becomes

$$\frac{x^2}{\lambda_1} + \frac{y^2}{\lambda_2} + \frac{z^2}{\lambda_3} = 0$$

Its sections with the imaginary ⊙le at infinity are given by

$$\frac{x^2}{\lambda_1} + \frac{y^2}{\lambda_2} + \frac{z^2}{\lambda_3} - \mu(x^2+y^2+z^2) = 0$$

This will break up into <u>real</u> factors

if $\mu = \frac{1}{\lambda_2}$ where λ_2 is mean between $\lambda_1\ \lambda_3$

And the sections are given by

$$x^2\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right) = z^2\left(\frac{1}{\lambda_3} - \frac{1}{\lambda_2}\right).$$

These two planes, pass through the axis of y, — which is the normal at (α, β, γ) to the λ_2— confocal.

And we have seen at the outset that it is the hyperboloid of one sheet.

These two planes cut the ellipsoid into two real sections, which have each imaginary tangents drawn from that pt (α, β, γ) to the ⊙le at infinity.

∴ The point must be the focus of each.

To take a rather more general Problem let us find the foci of the section of the ellipsoid $\Sigma x^2/a^2 = 1$

by $\Sigma lx - p = 0$

In the 1st place if $(\alpha, \beta \gamma)$ were a focus, it must lie in the plane

$$\therefore \quad \Sigma l\alpha - p = 0 \quad \cdot \quad \cdot \qquad \mathrm{I}.$$

Again the enveloping cone to the ellipsoid with vertex at $(\alpha, \beta \gamma)$ is

$$\left(\Sigma \frac{\alpha^2}{a^2} - 1\right)\left(\Sigma \frac{x^2}{a^2} - 1\right) = \left(\Sigma \frac{x\alpha}{a^2} - 1\right)^2.$$

To express the condition that (α, β, γ) is the focus, is to express the condition that the plane $lx + my + nz - p = 0$ cuts the cone in two imaginary lines going to the circular points at infinity in the plane.

The conditions that the plane cuts it in circular section may be obtained by as well taking

$$\left(\Sigma \frac{\alpha^2}{a^2} - 1\right)\left(\Sigma \frac{x^2}{a^2}\right) = \left(\Sigma \frac{\alpha x}{a^2}\right)^2$$

$$\& \quad \Sigma lx = 0$$

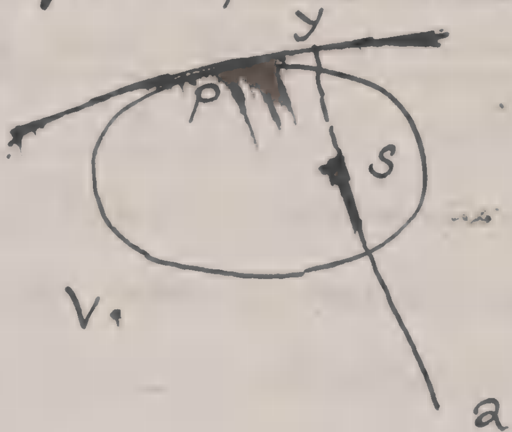
These conditions, which are two, together with

I give three equations to determine (α, β, γ).

Of course four sets of values will be obtained. Of these two give imaginary foci, & two are real ones.

——— o o o ———

2. To prove geometrically that, if a plane section be taken at right angles to either of the focal lines of a confocal system of cones, then the point of intersection is the focus of the section.



Let V be the vertex of the cone & P S Q Y be a section at ~~a right~~ right angles to VS, a focal line.

Then we can easily prove that such a plane gives polar section of the reciprocal cone.

Let SY ⊥ PY, a tangent plane to

the cone.

Then if yS produced backwards meets the reciprocal cone in Q

QV must be normal to the the tangent plane VPy.

$\therefore \angle QVy = \frac{\pi}{2}$.

Again $VS \perp yQ$ by hypothesis

($\because VS \perp$ pl $PSVQ$.

$\therefore Sy \cdot SQ = SV^2 =$ constant.

$\therefore$ The polar reciprocal of the locus of P, with respect to S, is the locus of Q which is a $\odot$le.

$\therefore$ S is the focus of the conic section which is the locus of P.

---

3. If P be any point on the spheric conic (intersection of a quadric cone with a concentric unit sphere) & S, H the foci, to prove that the great $\odot$lar arcs

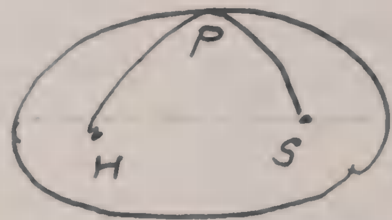
$PS + PH =$ Constant

P.T.O.

Let $\Sigma \frac{x^2}{a} = 0$ be the equation to the cone

then if $a > b > c$ the focal lines are given by

$$\frac{x^2}{a-b} = \frac{z^2}{b-c}.$$



∴ The direction cosines of the lines are

$\lambda, 0, \nu$

& $\lambda, 0, -\nu$

where $\lambda = \frac{\sqrt{a-b}}{\sqrt{a-c}}$ $\nu = \frac{\sqrt{b-c}}{\sqrt{a-c}}$.

Let l, m, n be the ~~generators of the~~ direction cosines of the generator through P.

Then denoting PH, PS by θ_1 θ_2 we get

$$\cos\theta_1 = l\lambda + n\nu$$
$$\cos\theta_2 = l\lambda - n\nu.$$

and of course $l^2 + m^2 + n^2 = 1$.

$$\therefore \quad l = \frac{\cos\theta_1 + \cos\theta_2}{2\lambda}$$
$$n = \frac{\cos\theta_1 - \cos\theta_2}{2\nu}$$

now since (l, m, n) is a generator of the cone

$$\therefore \quad \Sigma \frac{l^2}{a} = 0$$

$$\therefore \quad \frac{l^2}{a} + \frac{1 - l^2 - n^2}{b} + \frac{n^2}{c} = 0$$

Substituting the values of l, n just obtained we get

$$l^2\left(\frac{1}{a} - \frac{1}{b}\right) + n^2\left(\frac{1}{c} - \frac{1}{b}\right) + \frac{1}{b} = 0$$

$$\left(\frac{1}{a} - \frac{1}{b}\right)\frac{(\cos\theta_1 + \cos\theta_2)^2}{4\lambda^2} + \left(\frac{1}{c} - \frac{1}{b}\right)\frac{(\cos\theta_1 - \cos\theta_2)^2}{4\nu^2} + \frac{1}{b} = 0$$

i.e. $$-\frac{\cancel{a-b}}{a\cancel{b}} \cdot \frac{a-c}{4\cancel{(a-b)}}(\cos\theta_1 + \cos\theta_2)^2$$

$$+ \frac{\cancel{b-c}}{\cancel{b}c} \cdot \frac{a-c}{\cancel{(b-c)}4}(\cos\theta_1 - \cos\theta_2)^2 + \frac{1}{\cancel{b}} = 0$$

$$\therefore \quad \frac{(\cos\theta_1 + \cos\theta_2)^2}{a} - \frac{(\cos\theta_1 - \cos\theta_2)^2}{c} = \frac{4}{a-c}$$

$$\therefore (\cos\theta_1^2 + \cos\theta_2^2)\left(\frac{1}{a} - \frac{1}{c}\right) + 2\cos\theta_1 \cos\theta_2\left(\frac{1}{a} + \frac{1}{c}\right) = \frac{4}{a-c}$$

$$\therefore (\cos^2\theta_1 + \cos^2\theta_2) - 2\cos\theta_1\cos\theta_2\frac{a+c}{a-c} = -\frac{4ac}{(a-c)^2}$$

$$= 1 - \left(\frac{a+c}{a-c}\right)^2.$$

which gives

$$\sin^2\theta_1 \sin^2\theta_2 = \left(\cos\theta_1\cos\theta_2 - \frac{a+c}{a-c}\right)^2$$

extracting the square root we get

$$\cancel{\cos\theta_1 \cos\theta_2} = \cancel{\pm (\sin\theta_1}$$

$$\sin\theta_1 \sin\theta_2 = \pm \left(\cos\theta_1 \cos\theta_2 - \frac{a+c}{a-c} \right).$$

i.e. $$\cos\theta_1 \cos\theta_2 \pm \sin\theta_1 \sin\theta_2 = \pm \frac{a+c}{a-c}$$

$$\therefore \theta_1 \mp \theta_2 = \cos^{-1} \frac{a+c}{a-c}$$

now obviously in the present case, the difference of PH, PS could not be constants

$$\therefore \theta_1 + \theta_2 = \text{constant} = \cos^{-1} \frac{a+c}{a-c}.$$

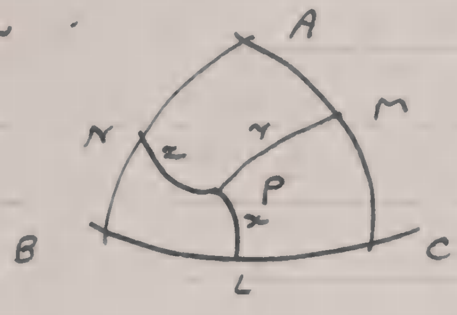
The negative sign is applicable when the diametrically opposite point is taken of one of the two foci.

It is obvious from ~~the~~ results that ~~one~~ of the two quantities a, c must be -ve for the existence of real cone.

A note on normal

Co-ordinates in a spherical triangle.

Let ABC be a spherical Δ & P any point from which great circular arcs are drawn to the sides of the Δ, perpendicular to them.



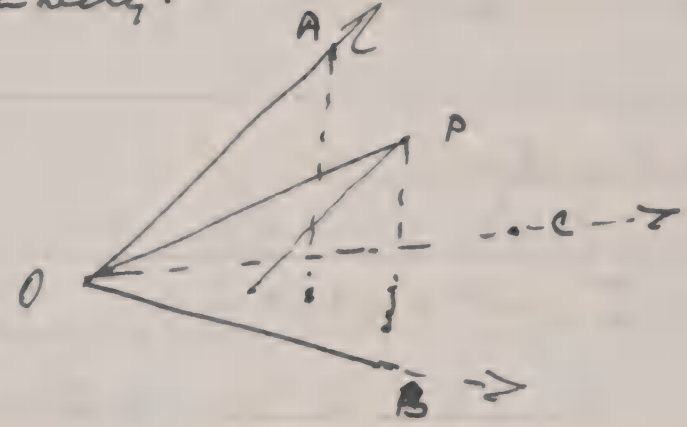
Then the sines of the arcs PL, PM PN can be taken as the Co-ordinates of the point P with regard to the Δ ABC.

$$x = \sin PL$$
$$y = \sin PM$$
$$z = \sin PN.$$

Then (x, y, z) are not independent of each other. But there must exist some relation between them to express the fact that the radius to the point P is equal to unity (Unit sphere is taken).

If we take the perpendicular arcs from A, B, C on the sides & call them $\sin p_1$, $\sin p_2$, $\sin p_3$, we see that if we use OA, OB, OC as co-ordinate axes (oblique),

then the ratios of x, y, z to $\sin p_1, \sin p_2, \sin p_3$ would be the ratios of perpendiculars from P, γ, A, B, C on the planes OBC, OCA, OAB respectively.



and since $OA = OB = OC = OP = 1$

$$\frac{x}{\sin p_1} = x\text{-co-ordinate of } P \text{ with regard to the oblique axes } OA, OB, OC.$$

Now $OP^2 = 1 = \xi^2 + \eta^2 + \zeta^2 + 2\Sigma\eta\zeta\cos\widehat{\eta\zeta}$

$\therefore$ Substituting we get

$$\Sigma \frac{x^2}{\sin^2 p_1} + 2\Sigma \frac{yz}{\sin p_2 \sin p_3}\cos a = 1.$$

which is the relation satisfied by the co-ordinates of any point P.

To put now in terms of the elements of

the triangle:—

$$\sin p_1 \sin a = \sin p_2 \sin b = \sin p_3 \sin c$$
$$= 2n = \sqrt{1 - \Sigma \cos^2 a - 2\Pi \cos a}$$
$$= 6 \text{ vol. } OABC. \qquad \{OA = = = 1.$$

∴ we get substituting for p's

$$\Sigma x^2 \sin^2 a + 2\Sigma yz \sin b \sin c \cos a$$
$$= 1 - \Sigma \cos^2 a - 2\Pi \cos a.$$

—— o o o o ——

To prove that the equation of any small circle can be thrown in the form

$$\Sigma \alpha^2 \sin^2 a + 2\Sigma \beta\gamma \sin b \sin c \cos a$$
$$= (l\alpha + m\beta + n\gamma)^2.$$

[Replacing x, y, z by α, β, γ].

The equation of a small circle is given by the intersection of a plane with sphere.

If the eq. to the sphere be

$$\Sigma \xi^2 + 2\Sigma \eta\zeta \cos \eta\zeta = OP^2 = 1$$

equation to any plane is $lx + my + nz = p$.

In normal co-ordinates both these become

$$\frac{\sum \alpha^2 \sin^2 a + 2\sum \beta\gamma \sin b \sin c \cos a}{\sqrt{1-\sum \cos^2 a + 2\Pi \cos a}} = \text{const.}$$

& $l\alpha + m\beta + n\gamma = \text{const.}$

Combining these two in a homogeneous form we see that, a small ⊙le is represented by

$$\sum \alpha^2 \sin^2 a + 2\sum \beta\gamma \sin b \sin c \cos a = (l\alpha + m\beta + n\gamma)^2$$

For the circum-⊙le points $(\sin p_1, 0, 0)$ $(0, \sin p_2, 0)$ & $(0, 0, \sin p_3)$ must lie on it.

i.e. $l = \sin a$ $m = \sin b$ $n = \sin c$

Substituting the equation at once reduces to the form

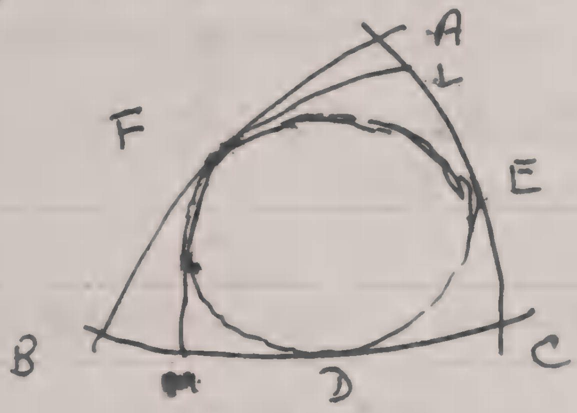
$$\sum \frac{\tan \frac{a}{2}}{\alpha} = 0$$

To find the equation of the in circle:—

The equation to any conic inscribed in the triangle must be of the form

$$\Sigma l^2 \alpha \pm 2 \Sigma mn \beta \gamma = 0$$

$\therefore$ if $\alpha = 0$ there must be one point given, as the circle touches &c.



If it be the incircle & touch the sides at D, E, F, as in the plane Δ we must have,

$$\left.\begin{aligned} AE &= AF = s-a \\ BD &= BF = s-b \\ \& \quad CD &= CE = s-c \end{aligned}\right\} 2s = a+b+c$$

If we put $\gamma = 0$ the equation becomes $l\alpha \pm m\beta = 0$

Let us choose $-$ve sign

Now $\alpha = \frac{FM}{\sin FM}$, $\beta = \cancel{FL}$ $\sin FL$

$\therefore\ l \sin FM = m \sin FL.$

But from the right angled Δ^s FAL, FBM

we get
$$FM = \sin(s-b)\sin B$$
$$FL = \sin(s-a)\sin A.$$

$$\therefore\ \frac{l}{\sin(s-a)\sin A} = \frac{m}{\sin(s-b)\sin B} = \text{by symmetry } \frac{n}{(s-c)\sin C}.$$

or since the sines of angles are proportional to the sines of sides,

$$\frac{l}{\sin a \sin(s-a)} = \frac{m}{\sin b \sin(s-b)} = \frac{n}{\sin c \sin(s-c)}.$$

Now we know that

$$\cos^2\frac{A}{2} = \frac{\sin s \sin(s-a)}{\sin b \sin c};\ \&c.$$

$$\therefore\ \frac{l}{\cos^2\frac{A}{2}} = \frac{m}{\cos^2\frac{B}{2}} = \frac{n}{\cos^2\frac{C}{2}}.$$

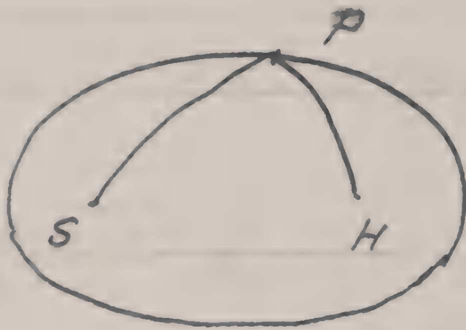
$\therefore$ The equation reduces to the form

$$\Sigma \cos^4 \frac{A}{2} \alpha^2 - 2 \Sigma \cos^2 \frac{B}{2} \cos^2 \frac{C}{2} \beta\gamma = 0$$

i.e. $\cos \frac{A}{2} \sqrt{\alpha} \pm \cos \frac{B}{2} \sqrt{\beta} \pm \cos \frac{C}{2} \sqrt{\gamma} = 0$

Q.E.D.

"The Focus and Directrix of Spher. Conic"



We have seen that the direction cosines of the focal lines VH, VS

are $\lambda, 0, \nu$

$\lambda \quad 0 \quad -\nu$

where $\lambda = \sqrt{\dfrac{a-b}{a-c}} \qquad \nu = \sqrt{\dfrac{b-c}{a-c}}$.

$\therefore$ If l, m, n be the direction cosines of any generator VP.

$$\sin^2 SP \; \cancel{\sin^2 SP} = 1 - (l\lambda + \cancel{m}^{\,n}\nu)^2$$

$$= (l^2+m^2+n^2) - l^2\,\frac{a-b}{a-c} - 2l\cancel{m}^{\,n}\,\frac{\sqrt{a-b}\sqrt{b-c}}{a-c}$$

$$- n^2\,\frac{b-c}{a-c}$$

And since the generator is of the cone

$$\Sigma \frac{x^2}{a} = 0$$

$$\therefore \Sigma \frac{l^2}{a} = 0$$

Eliminating m^2 we get

$$\sin^2 SP =$$

$$l^2 - b\left(\frac{l^2}{a} + \frac{n^2}{c}\right) + n^2$$

$$-l^2\frac{a-b}{a-c} - 2ln\frac{\sqrt{a-b}\cdot\sqrt{b-c}}{a-c}$$

$$-n^2\ \frac{b-c}{a-c}.$$

1st term =

$$= l^2\left(1 - \frac{b}{a} - \frac{a-b}{a-c}\right)$$

$$= (a-b)l^2\left[\frac{1}{a} - \frac{1}{a-c}\right] = -\frac{c(a-b)}{a(a-c)}l^2$$

Similarly, 3rd term =

$$\left(1 - \frac{b}{c} - \frac{b-c}{a-c}\right) = \frac{-a(b-c)}{c(a-c)}.$$

$$\therefore (\sin SP)^2 = \left\{l\frac{\sqrt{-c}}{\sqrt{a}}\sqrt{\frac{a-b}{a-c}} - \frac{n\sqrt{a}}{\sqrt{-c}}\sqrt{\frac{b-c}{a-c}}\right\}^2$$

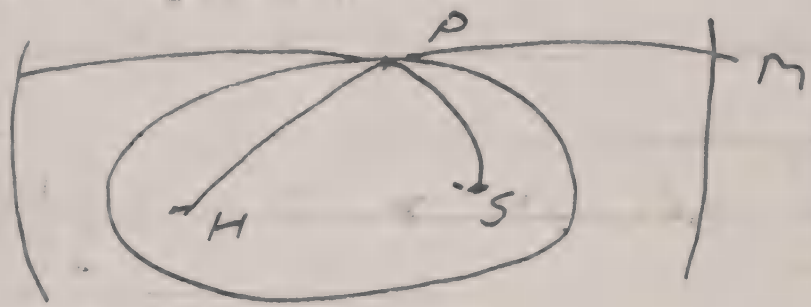
$$= (l\alpha + n\gamma)^2$$

Now if we take a line of which the direction cosines are $\alpha, 0\ \gamma$ or proportional to these,

It follows that

$\sin SP$ is proportional to the

sine of the angle which the generator $VP\ (l, m, n)$ makes with the plane $\perp$ line (α, β, γ).



i.e. $\sin^2 SP = e^2 \sin^2 PM$.

which is just analogue to the focus directrix property in the plane geometry.

Q.E.D.

Conicoids.

The general equation of a conicoid is

$$(a\ b\ d\ f\ g\ h\ u\ v\ w)(x\ y\ z\ 1)^2 = 0$$

We have proved that for any transformation of Axes in which the Axes remain rectangular

$$a+b+c,\ \Sigma bc-f^2 \ \&\ D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

are *invariants*.

We proceed to prove now that the expression,

$$\Delta \equiv \begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix}$$

(≡ usually called δ)

is an invariant for any transformation of Axes.

Proof:— The equation to the conicoid in its expanded form can be written as:—

$$ax^2 + by^2 + cz^2 + dt^2$$
$$+ 2fyz + 2gzx + 2hxy$$
$$+ 2uxt + 2vyt + 2zwt \quad = 0 \qquad (t=1).$$

And suppose now that the transformation is such that

$$x = l_1 x' + l_2 y' + l_3 z' + l_4 t'$$
$$y = m_1 x' + m_2 y' + m_3 z' + m_4 t'$$
$$z = n_1 x' + n_2 y' + n_3 z' + n_4 t'$$
$$t = p_1 x' + p_2 y' + p_3 z' + p_4 t'$$

Making these substitutions, the equation to the conicoid will be, say

$$(a'\ b'\ c'\ d'\ f'\ g'\ h'\ u'\ v'\ w' \between x\ y\ z\ t)^2 = 0$$

Picking out the coefficient of x^2, we get

$$a' = al_1^2 + bm_1^2 + cn_1^2 + dp_1^2 + 2fm_1n_1 + 2gn_1l_1$$
$$+ 2hl_1m_1 + 2ul_1p_1 + 2m_1vp_1 + 2wn_1p_1.$$

$$= l_1(al_1 + hm_1 + gn_1 + up_1)$$
$$+ m_1(\qquad)$$
$$+ n_1(\qquad)$$
$$+ p_1(\qquad)$$

In the same way the coefficient of $2xy$ =

b' = the same as a', with $l_2\, m_2\, n_2\, p_2$ substituted for $l_1\, m_1\, n_1\, p_1$ outside the brackets.

Now consider the result of multiplying the three determinants:

$$\begin{vmatrix} l_1 & m_1 & n_1 & p_1 \\ l_2 & m_2 & n_2 & p_2 \\ l_3 & m_3 & n_3 & p_3 \\ l_4 & m_4 & n_4 & p_4 \end{vmatrix} \begin{vmatrix} l_1 & m_1 & n_1 & p_1 \\ l_2 & m_2 & n_2 & p_2 \\ l_3 & m_3 & n_3 & p_3 \\ l_4 & m_4 & n_4 & p_4 \end{vmatrix} \begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix}$$

It is easy to see from the forms we have written down that the product is just equal to

$$\begin{vmatrix} a' & h' & g' & u' \\ h' & b' & f' & v' \\ g' & f' & c' & w' \\ u' & v' & w' & d' \end{vmatrix}$$

This is a perfectly general result.

Applying this to the particular

case of geometry we must take $(l_1\ m_1\ n_1)\ (l_2\ m_2\ n_2)\ (l_3\ m_3\ n_3)$ in such a way that they refer to three axes at <u>right</u> <u>angles</u>.

And for $l_4\ n_4\ m_4$ ~~t~~ we must take quantities $\alpha\ \beta\ \gamma\ \delta$ such that they are results obtained by change of origin.

And the last line becomes

$$t = 0 + 0 + 0 + t'$$

Then the

$$\begin{vmatrix} l_1 & m_1 & n_1 & \alpha \\ & & & \\ 0 & 0 & 0 & t' \end{vmatrix} \quad \begin{vmatrix} l_1 & m_1 & n_1 & \alpha \\ & & & \\ 0 & 0 & 0 & t' \end{vmatrix}$$

becomes simply

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}^2$$

$$= (\pm 1)^2$$

$$= 1$$

∴ And thus we see

$$\Delta = \begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix} = \begin{vmatrix} a' & h' & g' & u' \\ h' & b' & f' & v' \\ g' & f' & c' & w' \\ u' & v' & w' & d' \end{vmatrix}$$

or, Δ is an invariant.

Q.E.D.